

Divide and Congruence II: From Decomposition of Modal Formulas to Preservation of Delay and Weak Bisimilarity

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Earlier we presented a method to decompose modal formulas for processes with the internal action τ , and congruence formats for branching and η -bisimilarity were derived on the basis of this decomposition method. The idea is that a congruence format for a semantics must ensure that the formulas in the modal characterisation of this semantics are always decomposed into formulas that are again in this modal characterisation. In this follow-up paper the decomposition method is enhanced to deal with modal characterisations that contain a modality $\langle \varepsilon \rangle a \phi$, to derive congruence formats for delay and weak bisimilarity.

1 Introduction

In [2] a method was developed to generate congruence formats for (concrete) process semantics from their modal characterisation. It crosses the borders between process algebra, structural operational semantics, process semantics, and modal logic. Cornerstone is the work in [19] to decompose formulas from Hennessy-Milner logic [17] with respect to a structural operational semantics in the De Simone format [23]. It was extended to the ntyft format [15] without lookahead in [2], and to the tyft format [16] in [8].

An equivalence is a congruence for a given process algebra or programming language if the equivalence class of a term $f(p_1, \dots, p_n)$ is determined by the function f and the equivalence classes of its arguments p_1, \dots, p_n . Being a congruence is an important property, for instance to fit a process semantics into an axiomatic framework. A wide range of syntactic formats for structural operational semantics have been developed for several process semantics, to ensure that such a semantics is a congruence; notably for unrooted and rooted weak bisimilarity in [1] and for unrooted and rooted delay bisimilarity in [14]. These formats are contained in the positive GSOS format [3]. They include so-called patience rules for arguments i of function symbols f , which imply that any term $f(p_1, \dots, p_n)$ inherits the τ -transitions of its argument p_i .

Key idea in [2] is that a congruence format for a process semantics must ensure that the formulas in a modal characterisation of this semantics are always decomposed into formulas that are again in this modal characterisation. This yielded congruence formats for all known concrete process semantics in a convenient way. Moreover, the resulting congruence formats are more elegant and expressive than existing congruence formats for individual process semantics. In [9] this method was extended to weak

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process semantics, which take into account the internal action τ . As a result, congruence formats for rooted branching and η -bisimilarity were derived. These formats use two predicates \aleph and Λ on arguments of function symbols: \aleph marks processes that can execute immediately, and Λ marks processes that have started executing (but may currently be unable to execute). Formats for unrooted branching and η -bisimilarity were obtained by imposing one extra restriction on top of the format for the corresponding rooted semantics: Λ holds universally.

The framework from [9] covers only a small part of the spectrum of weak semantics from [12]. In particular, it does not readily extend to delay and weak bisimilarity [20, 21]. The reason is that in the definition of these semantics, in contrast to branching and η -bisimilarity, a process q that mimics an a -transition from a process p , does not need to be related to p at the moment that q performs the a -transition. This implies that in the modal characterisation of delay and weak bisimilarity, a modality $\langle a \rangle \varphi$ stating that an a -transition to a process where φ holds, is always preceded by a modality $\langle \varepsilon \rangle$ allowing any number of τ -transitions. As a consequence, devising congruence formats for delay and weak bisimilarity has been notoriously difficult, see e.g. [1, 14]. Here we show how this technical obstacle can be overcome by means of the semantic notion of *delay resistance*, which generalises earlier notions from [1, 14]. This notion ensures that modalities $\langle \varepsilon \rangle \langle a \rangle \varphi$ are decomposed into formulas that again have this form. Thus congruence formats can be derived for semantics with a modal characterisation containing such modalities. We derive congruence formats for rooted delay and weak bisimilarity. The congruence formats for the unrooted counterparts of these semantics are again obtained by the extra requirement that Λ must be universal. We moreover provide syntactic restrictions which imply delay resistance, leading to the first entirely syntactic congruence formats for rooted delay and weak bisimilarity.

In [14] a general method is presented to generalise any congruence format F , contained in the GSOS format, into a *two-tiered* version of F . Two-tiered formats distinguish so-called “principal” function symbols and “abbreviations”. The latter can be regarded as syntactic sugar, adding nothing that could not be expressed with principal function symbols. The original format F is essentially the restriction of its two-tiered version that allows principal function symbols only. As shown in [14], the general formats of [1, 14] can be obtained as the two-tiered versions of the simplified formats from [1, 14]. In [9] this two-tiered approach was generalised from GSOS to decent ntyft format. Consequently, the two-tiered versions of the congruence formats presented in the current paper are again congruence formats for rooted/unrooted delay/weak bisimilarity. These two-tiered versions of our formats (or more precisely, of the intersection of our formats with the decent ntyft format) generalise the full formats of [1, 14]. Ulidowski [24, 25, 26] proposed congruence formats for weak semantics with a different treatment of divergence, which interestingly allow the inclusion of the priority operator; (divergence-insensitive) rooted weak bisimilarity is not a congruence for this operator. The TSSs of $\text{BPA}_{\varepsilon\delta\tau}$, binary Kleene star and deadlock testing in Sect. 6 are however outside those formats.

This research line shows that it is worthwhile to study the interplay of structural operational semantics and modal logic. The modal characterisation of a process semantics turns out to be fundamental for its congruence properties. Although some rather heavy technical machinery is needed to set the scene, especially the derivation of so-called ruloids and the decomposition method for modal formulas, the bulk of this work can be reused for the development of congruence formats for other weak process semantics. This is witnessed by the fact that the congruence results for rooted delay and weak bisimilarity are obtained in an almost identical fashion, and build upon the congruence proofs for rooted branching bisimilarity in [9]. Furthermore, the congruence formats that we obtain here are more liberal and more elegant than existing congruence formats for these semantics. In particular, in [1] it is stated that the RWB format put forward in that paper has a “horrible definition”. In [1] it is moreover stated that “negative rules seem incompatible with weak process equivalences.” Here we show how negative premises can be

included in congruence formats for rooted delay and weak bisimilarity.

The paper is structured as follows. Sect. 2 contains technical preliminaries. Sect. 3 introduces the notion of delay resistance and explains how the decomposition method of modal formulas from [9] needs to be adapted. Sect. 4 presents the congruence formats for rooted delay and weak bisimilarity and the proofs of these congruence results. Sect. 5 shows that it is sufficient to check delay resistance for a limited set of variables (to be precise, the Λ -frozen arguments of a source), and how the semantic notion of delay resistance can be captured by means of syntactic criteria. Sect. 6 provides applications of our congruence formats. Sect. 7 concludes the paper.

2 Preliminaries

This section recalls the basic notions of labelled transition systems and weak semantics (Sect. 2.1), and presents modal characterisations of the semantic equivalences that are studied in this paper (Sect. 2.2). Then follows a brief introduction to structural operational semantics and the notion of a well-supported proof (Sect. 2.3). Next we recall some syntactic restrictions on transition rules (Sect. 2.4). Then we present the notion of patience rules (Sect. 2.5), and a basic result from [2], Prop. 1, regarding ruloids (Sect. 2.6). Sect. 2.7 shows that in Prop. 1 we may restrict attention to ruloids with so-called linear proofs. Finally, we recall from [9] a method for decomposition of modal formulas (Sect. 2.8).

2.1 Equivalences on labelled transition systems

A *labelled transition system (LTS)* is a pair $(\mathbb{P}, \rightarrow)$, with \mathbb{P} a set of *processes* and $\rightarrow \subseteq \mathbb{P} \times (A \cup \{\tau\}) \times \mathbb{P}$, where τ is an *internal action* and A a set of *concrete actions* not containing τ . We use p, q to denote processes, α, β, γ for elements of $A \cup \{\tau\}$, and a, b for elements of A . We write $p \xrightarrow{\alpha} q$ for $(p, \alpha, q) \in \rightarrow$ and $p \not\xrightarrow{\alpha}$ for $\neg(\exists q \in \mathbb{P} : p \xrightarrow{\alpha} q)$. Furthermore, $\xRightarrow{\tau}$ denotes the transitive-reflexive closure of $\xrightarrow{\tau}$.

Processes can be distinguished from each other by a wide range of semantics, based on e.g. branching structure or decorated versions of execution sequences. VAN GLABBEK [12] classified so-called weak semantics, which take into account the internal action τ . Here we focus on two such equivalences which, to different degrees, abstract away from internal actions: delay bisimilarity [20] and weak bisimilarity [21]. They are the two weak semantics that are employed most widely in the literature.

Definition 1 Let $B \subseteq \mathbb{P} \times \mathbb{P}$ be a symmetric relation.

- B is a *delay bisimulation* if pBq and $p \xrightarrow{\alpha} p'$ implies that either $\alpha = \tau$ and $p'Bq$, or $q \xRightarrow{\tau} \xrightarrow{\alpha} q''$ for some q'' with $p'Bq''$.

Processes p, q are *delay bisimilar*, denoted $p \stackrel{\text{d}}{\sim} q$, if there exists a delay bisimulation B with pBq .

- B is a *weak bisimulation* if pBq and $p \xrightarrow{\alpha} p'$ implies that either $\alpha = \tau$ and $p'Bq$, or $q \xRightarrow{\tau} \xrightarrow{\alpha} \xRightarrow{\tau} q''$ for some q'' with $p'Bq''$.

Processes p, q are *weakly bisimilar*, denoted $p \stackrel{\text{w}}{\sim} q$, if there exists a weak bisimulation B with pBq .

The notions of delay and weak bisimilarity were originally both introduced by Milner under the name “observation equivalence”. Clearly, delay bisimilarity is included in weak bisimilarity.

It is well-known that delay and weak bisimilarity constitute equivalence relations [20, 21]. However, these two semantics are not *congruences* for most process algebras from the literature, meaning that the equivalence class of a process $f(p_1, \dots, p_n)$, with f an n -ary function symbol, is not always determined

by the equivalence classes of its arguments, i.e. the processes p_1, \dots, p_n . Rooted counterparts of these equivalences were introduced, which require for the pair of initial states that a τ -transition needs to be matched by at least one τ -transition. Unlike the unrooted versions they are congruences for basic process algebras, notably for the alternative composition operator.

Definition 2 Let $R \subseteq \mathbb{P} \times \mathbb{P}$ be a symmetric relation.

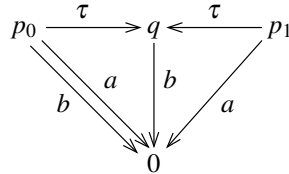
- R is a *rooted delay bisimulation* if pRq and $p \xrightarrow{\alpha} p'$ implies that $q \xRightarrow{\varepsilon} \xrightarrow{\alpha} q'$ for some q' with $p' \xleftrightarrow{d} q'$.

Processes p, q are *rooted delay bisimilar*, denoted $p \xleftrightarrow{rd} q$, if there exists a rooted delay bisimulation R with pRq .

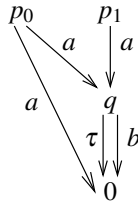
- R is a *rooted weak bisimulation* if pRq and $p \xrightarrow{\alpha} p'$ implies that $q \xRightarrow{\varepsilon} \xrightarrow{\alpha} \xRightarrow{\varepsilon} q'$ for some q' with $p' \xleftrightarrow{w} q'$.

Processes p, q are *rooted weakly bisimilar*, denoted $p \xleftrightarrow{rw} q$, if there exists a rooted weak bisimulation R with pRq .

Example 1 The processes p_0 and p_1 in the following LTS are rooted delay bisimilar but not η -bisimilar. The idea is that in an η -bisimulation the transition $p_0 \xrightarrow{b} 0$ cannot be mimicked by p_1 ; the only candidate $p_1 \xrightarrow{\tau} q \xrightarrow{b} 0$ fails because q cannot be related to p_0 , while this would be required for an η -bisimulation.



The processes p_0 and p_1 in the following LTS are rooted weakly bisimilar but not delay bisimilar. The idea is that in a delay bisimulation the transition $p_0 \xrightarrow{a} 0$ cannot be mimicked by p_1 ; the only candidate $p_1 \xrightarrow{a} q \xrightarrow{\tau} 0$ fails because q cannot be related to 0 , while this would be required for a delay bisimulation.



Our main aim is to develop congruence formats for both the rooted and the unrooted versions of the two weak semantics defined in this section. These congruence formats will impose syntactic restrictions on the transition rules (see Sect. 2.3) that are used to generate the underlying LTS. The congruence formats will be determined using the characterising modal logics for these two weak semantics, which are presented in the next section.

2.2 Modal logic

Behavioural equivalences can be characterised in terms of the observations that an experimenter could make during a session with a process. Modal logic captures such observations, with the aim to formulate properties of processes in an LTS. Following [12], we extend Hennessy-Milner logic [17] with the modal connective $\langle \varepsilon \rangle \varphi$, expressing that a process can perform zero or more τ -transitions to a process where φ holds.

Definition 3 The class \mathbb{O} of *modal formulas* is defined as follows, where I ranges over all index sets:

$$\mathbb{O} \quad \varphi ::= \bigwedge_{i \in I} \varphi_i \mid \neg \varphi \mid \langle \alpha \rangle \varphi \mid \langle \varepsilon \rangle \varphi .$$

$p \models \varphi$ denotes that p satisfies φ . By definition, $p \models \langle \alpha \rangle \varphi$ if $p \xrightarrow{\alpha} p'$ for some p' with $p' \models \varphi$, and $p \models \langle \varepsilon \rangle \varphi$ if $p \xRightarrow{\varepsilon} p'$ for some p' with $p' \models \varphi$. We use abbreviations \top for the empty conjunction and $\varphi_1 \wedge \varphi_2$ for $\bigwedge_{i \in \{1,2\}} \varphi_i$. We write $\varphi \equiv \varphi'$ if $p \models \varphi \Leftrightarrow p \models \varphi'$ for any process p in any LTS.

A modal characterisation of an equivalence on processes consists of a class C of modal formulas such that two processes are equivalent if and only if they satisfy the same formulas in C . Hennessy-Milner logic is a modal characterisation of bisimilarity. We now introduce modal characterisations for (unrooted and rooted) delay and weak bisimilarity.

Definition 4 The subclasses \mathbb{O}_e and \mathbb{O}_{re} of \mathbb{O} , for $e \in \{d, w\}$, are defined as follows:

$$\begin{array}{ll} \mathbb{O}_d & \varphi ::= \bigwedge_{i \in I} \varphi_i \mid \neg \varphi \mid \langle \varepsilon \rangle \varphi \mid \langle \varepsilon \rangle \langle a \rangle \varphi \\ \mathbb{O}_{rd} & \varphi ::= \bigwedge_{i \in I} \varphi_i \mid \neg \varphi \mid \langle \varepsilon \rangle \langle \alpha \rangle \hat{\varphi} \mid \hat{\varphi} \ (\hat{\varphi} \in \mathbb{O}_d) \\ \\ \mathbb{O}_w & \varphi ::= \bigwedge_{i \in I} \varphi_i \mid \neg \varphi \mid \langle \varepsilon \rangle \varphi \mid \langle \varepsilon \rangle \langle a \rangle \langle \varepsilon \rangle \varphi \\ \mathbb{O}_{rw} & \varphi ::= \bigwedge_{i \in I} \varphi_i \mid \neg \varphi \mid \langle \varepsilon \rangle \langle \alpha \rangle \langle \varepsilon \rangle \hat{\varphi} \mid \hat{\varphi} \ (\hat{\varphi} \in \mathbb{O}_w). \end{array}$$

In these definitions, a ranges over A and α over $A \cup \{\tau\}$. The classes \mathbb{O}_e^{\equiv} and \mathbb{O}_{re}^{\equiv} denote the closures of \mathbb{O}_e , respectively \mathbb{O}_{re} , under \equiv .

The last clause in the definition of \mathbb{O}_{re} guarantees that $\mathbb{O}_e \subset \mathbb{O}_{re}$, which will be needed in the proof of Prop. 5. If this clause were omitted, it would still follow that $\mathbb{O}_e^{\equiv} \subset \mathbb{O}_{re}^{\equiv}$, using structural induction together with $\langle \varepsilon \rangle \varphi \equiv \varphi \vee \langle \varepsilon \rangle \langle \tau \rangle \varphi$ (for $e = d$) or $\langle \varepsilon \rangle \varphi \equiv \varphi \vee \langle \varepsilon \rangle \langle \tau \rangle \langle \varepsilon \rangle \varphi$ (for $e = w$).

For $L \subseteq \mathbb{O}$, we write $p \sim_L q$ if p and q satisfy the same formulas in L . Note that, trivially, $p \sim_{\mathbb{O}_e} q \Leftrightarrow p \sim_{\mathbb{O}_e^{\equiv}} q$ and $p \sim_{\mathbb{O}_{re}} q \Leftrightarrow p \sim_{\mathbb{O}_{re}^{\equiv}} q$.

Theorem 1 $p \stackrel{\varepsilon}{\leftrightarrow}_e q \Leftrightarrow p \sim_{\mathbb{O}_e} q$ and $p \stackrel{\varepsilon}{\leftrightarrow}_{re} q \Leftrightarrow p \sim_{\mathbb{O}_{re}} q$, for all $p, q \in \mathbb{P}$, where $e \in \{d, w\}$.

The first statement is a restatement of standard results from [17, 20, 21, 12]; the second statement is an easy corollary, obtained in the same vein. Moreover, a proof for the case $e = w$ is presented in Appendix A. The proof for the case $e = d$ is similar.

2.3 Structural operational semantics

A *signature* is a set Σ of function symbols f with arity $ar(f)$. Let V be an infinite set of variables, with typical elements x, y, z ; we always take $|\Sigma|, |A| \leq |V|$. A syntactic object is *closed* if it does not contain any variables. The set $\mathbb{T}(\Sigma)$ of terms over Σ and V is defined as usual; t, u, v, w denote terms and $var(t)$

is the set of variables that occur in term t . A term is *univariate* if it is without multiple occurrences of the same variable. A substitution σ is a partial function from V to $\mathbb{T}(\Sigma)$. A closed substitution is a total function from V to closed terms. The domain of substitutions is extended to $\mathbb{T}(\Sigma)$ as usual.

Structural operational semantics [22] provides process algebras and specification languages with an interpretation. It generates an LTS, in which processes are the closed terms over a (single-sorted, first-order) signature, and transitions between processes may be supplied with labels. The transitions between processes are obtained from a transition system specification, which consists of a set of proof rules called transition rules.

Definition 5 A (*positive* or *negative*) *literal* is an expression $t \xrightarrow{\alpha} u$ or $t \xrightarrow{\alpha} \cdot$. A (*transition*) *rule* is of the form $\frac{H}{\lambda}$ with H a set of literals called the *premises*, and λ a literal called the *conclusion*; the term at the left-hand side of λ is called the *source* of the rule. Given a transition rule $\frac{H}{\lambda}$, write H^+ for the set of positive premises in H , and H^{s-} for the set of *stable* negative premises in H : those premises $t \xrightarrow{\alpha} \cdot$ for which also the premise $t \xrightarrow{\alpha} \cdot$ is in H . With $rhs(H)$ we denote the set of right-hand sides of the premises in H^+ . A rule $\frac{\emptyset}{\lambda}$ is also written λ . A rule is *standard* if it has a positive conclusion, and *positive* if moreover it has only positive premises. A *transition system specification* (TSS), written (Σ, R) , consists of a signature Σ and a collection R of transition rules over Σ . A TSS is *standard* or *positive* if all its rules are.

The following definition tells when a literal is provable from a TSS. It generalises the standard definition (see e.g. [16]) by allowing the derivation of transition rules. The derivation of a literal λ corresponds to the derivation of the transition rule $\frac{H}{\lambda}$ with $H = \emptyset$. The case $H \neq \emptyset$ corresponds to the derivation of λ under the assumptions H .

Definition 6 Let $P = (\Sigma, R)$ be a TSS. An *irredundant proof* from P of a transition rule $\frac{H}{\lambda}$ is a well-founded tree with the nodes labelled by literals and some of the leaves marked ‘‘hypothesis’’, such that the root has label λ , H is the set of labels of the hypotheses, and if μ is the label of a node that is not a hypothesis and K is the set of labels of the children of this node then $\frac{K}{\mu}$ is a substitution instance of a transition rule in R .

The proof of $\frac{H}{\lambda}$ is called *irredundant* [2] because H must equal (instead of include) the set of labels of the hypotheses. Irredundancy will be crucial for the preservation under provability of our congruence formats; see Sect. 4.2. Namely, in a ‘redundant’ proof one can freely add premises to the derived rule, so also a premise that violates a syntactic restriction of the congruence format under consideration.

A TSS is meant to specify an LTS in which the transitions are closed positive literals. A standard TSS with only positive premises specifies an LTS in a straightforward way, but it is not so easy to associate an LTS to a TSS with negative premises. From [13] we adopt the notion of a well-supported proof of a closed literal. Literals $t \xrightarrow{\alpha} u$ and $t \xrightarrow{\alpha} \cdot$ are said to *deny* each other.

Definition 7 Let $P = (\Sigma, R)$ be standard TSS. A *well-supported proof* from P of a closed literal λ is a well-founded tree with the nodes labelled by closed literals, such that the root is labelled by λ , and if μ is the label of a node and K is the set of labels of the children of this node, then:

1. either μ is positive and $\frac{K}{\mu}$ is a closed substitution instance of a transition rule in R ;
2. or μ is negative and for each set N of closed negative literals with $\frac{N}{\nu}$ irredundantly provable from P and ν a closed positive literal that denies μ , a literal in K denies one in N .

$P \vdash_{ws} \lambda$ denotes that a well-supported proof from P of λ exists. A standard TSS P is *complete* if for each p and α , either $P \vdash_{ws} p \xrightarrow{\alpha}$ or $P \vdash_{ws} p \xrightarrow{\alpha} p'$ for some p' .

In [13] it was shown that \vdash_{ws} is consistent, in the sense that no standard TSS admits well-supported proofs of two literals that deny each other. A complete TSS specifies an LTS, consisting of the *ws*-provable closed positive literals.

2.4 Syntactic restrictions on transition rules

In this section we present terminology for syntactic restrictions on rules, originating from [2, 15, 16], where congruence formats are presented for a range of concrete semantics (which do not take into account the internal action τ).

Definition 8 An *ntytt rule* is a transition rule in which the right-hand sides of positive premises are variables that are all distinct, and that do not occur in the source. An *ntytt rule* is an *ntyxt rule* if its source is a variable, an *ntyft rule* if its source contains exactly one function symbol and no multiple occurrences of variables, an *nxytt rule* if the left-hand sides of its premises are variables, and an *xyntt rule* if the left-hand sides of its positive premises are variables. An *xynft rule* is both *ntyft* and *xyntt*.

Definition 9 A variable in a transition rule is *free* if it occurs neither in the source nor in right-hand sides of premises. A transition rule has *lookahead* if some variable occurs in the right-hand side of a premise and in the left-hand side of a premise. A transition rule is *decent* if it has no lookahead and does not contain free variables.

Each combination of syntactic restrictions on transition rules induces a corresponding syntactic format for TSSs of the same name. For instance, a TSS is in *decent ntyft* format if it contains decent *ntyft* rules only.

The following lemma, on the preservation of decency under irredundant provability, was proved in [2].

Lemma 1 Let P be a TSS in decent *ntytt* format. Then any *ntytt* rule irredundantly provable from P is decent.

We define two more syntactic formats for TSSs. The *ntyft/ntyxt* and *ready simulation* formats [15, 2] were originally introduced to guarantee congruence for bisimilarity and ready simulation.

Definition 10 A TSS is in *ntyft/ntyxt format* if it consists of *ntyft* and *ntyxt* rules, and in *ready simulation format* if moreover its transition rules have no lookahead.

2.5 Patience rules

We introduce some terminology for predicates on arguments of function symbols from [1, 2].

Definition 11 Let Γ be a unary predicate on $\{(f, i) \mid 1 \leq i \leq ar(f), f \in \Sigma\}$. If $\Gamma(f, i)$, then argument i of f is Γ -*liquid*; otherwise it is Γ -*frozen*. An occurrence of x in t is Γ -*liquid* if either $t = x$, or $t = f(t_1, \dots, t_{ar(f)})$ and the occurrence is Γ -*liquid* in t_i for a liquid argument i of f ; otherwise the occurrence is Γ -*frozen*.

Note that an occurrence of a variable x in a term $t \in \mathbb{T}(\Sigma)$ is Γ -frozen if and only if t contains a subterm $f(t_1, \dots, t_{ar(f)})$ such that the occurrence of x is in t_i for a Γ -frozen argument i of f .

In Sect. 2.8 we will present a method for decomposing modal formulas that gives a special treatment to arguments of function symbols that are deemed *patient*; we will use a predicate Γ to mark the arguments that get this special treatment.

Definition 12 [1, 5] A standard ntyft rule is a *patience rule* for argument i of f if it is of the form

$$\frac{x_i \xrightarrow{\tau} y}{f(x_1, \dots, x_{ar(f)}) \xrightarrow{\tau} f(x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_{ar(f)})}$$

Given a predicate Γ , the rule above is called a Γ -*patience rule*, if $\Gamma(f, i)$. A TSS is Γ -*patient* if it contains all Γ -patience rules. A standard ntytt rule is Γ -*patient* if it is irredundantly provable from the Γ -patience rules; else it is called Γ -*impatient*.

A patience rule for an argument i of a function symbol f expresses that terms $f(p_1, \dots, p_n)$ can mimic the τ -transitions of argument p_i . Typically, in process algebra, there are patience rules for both arguments of the merge operator and for the first argument of sequential composition, as they can contain running processes, but not for the arguments of alternative composition or for the second argument of sequential composition.

2.6 Ruloids

To decompose modal formulas, we use a result from [2], where for any standard TSS P in ready simulation format a collection of decent nxytt rules, called P -*ruloids*, is constructed. We explain this construction at a rather superficial level; the precise transformation can be found in [2].

First P is converted to a standard TSS P^\dagger in decent ntyft format. In this conversion from [16], free variables in a rule are replaced by arbitrary closed terms, and if the source is of the form x , then this variable is replaced by a term $f(x_1, \dots, x_{ar(f)})$ for each function symbol f in the signature of P , where the variables $x_1, \dots, x_{ar(f)}$ are fresh.

Next, using a construction from [6], left-hand sides of positive premises are reduced to variables. Roughly the idea is, given a premise $f(t_1, \dots, t_n) \xrightarrow{\alpha} y$ in a rule r , and another rule $\frac{H}{f(x_1, \dots, x_n) \xrightarrow{\alpha} t}$, to transform r by replacing the aforementioned premise by H , y by t , and the x_i by the t_i ; this is repeated (transfinitely) until all positive premises with a non-variable term as left-hand side have disappeared. This yields an intermediate standard TSS P^\ddagger in xynft format, of which all the rules are irredundantly provable from P . In fact, the rules of P^\ddagger are exactly the xynft rules irredundantly provable from P^\dagger . The motivation for this transformation step is that for TSSs in xynft format the semantic phrase “for each set N of closed negative literals with $\frac{N}{V}$ irredundantly provable from P ” in the second clause of Def. 7 of a well-supported proof can be replaced by a syntactic phrase: “for each closed substitution instance $\frac{N}{V}$ of a rule in R ”.

In the final transformation step, non-standard rules with a negative conclusion $t \xrightarrow{\alpha}$ are introduced. The motivation is that instead of the notion of well-founded provability of Def. 7, we want a more constructive notion like Def. 6, by making it possible that a negative premise is matched with a negative conclusion. A non-standard rule $\frac{H}{f(x_1, \dots, x_n) \xrightarrow{\alpha} t}$ is obtained by picking one premise from each xynft rule in P^\ddagger with a conclusion of the form $f(x_1, \dots, x_n) \xrightarrow{\alpha} t$, and including the denial of each of the selected premises as a premise in H .

The resulting TSS, which is in decent ntyft format, is denoted by P^+ . The above construction implies that if P is Γ -patient, then so is P^+ . In [2] it was established, for all closed literals μ , that $P \vdash_{ws} \mu$ if and only if μ is irredundantly provable from P^+ . By definition, the P -ruloids are the (decent) nxytt rules irredundantly provable from P^+ .

The following correspondence result from [2] between a TSS and its ruloids is crucial for the soundness of the decomposition method presented in Sect. 2.8. It says that there is a well-supported proof from P of a transition $\rho(t) \xrightarrow{a} q$, with ρ a closed substitution, if and only if there is a derivation of this transition that uses at the root a P -ruloid with source t .

Proposition 1 Let $P = (\Sigma, R)$ be a standard TSS in ready simulation format, $t \in \mathbb{T}(\Sigma)$ and $\rho : V \rightarrow \mathbb{T}(\Sigma)$ a closed substitution. Then $P \vdash_{ws} \rho(t) \xrightarrow{\alpha} q$ if and only if there are a P -ruloid $\frac{H}{t \xrightarrow{\alpha} u}$ and a closed substitution ρ' such that $P \vdash_{ws} \rho'(\mu)$ for all $\mu \in H$, $\rho'(t) = \rho(t)$ and $\rho'(u) = q$.

2.7 Linear proofs

Definition 13 An irredundant proof of a transition rule $\frac{H}{\lambda}$ is called *linear* if no two hypotheses in the proof tree of Def 6 are labelled with the same positive premise.

Example 2 From the TSS with rules

$$\frac{x \xrightarrow{a} y}{f(x) \xrightarrow{b} x} \quad \frac{x \xrightarrow{a} y}{f(x) \xrightarrow{c} x} \quad \frac{x \xrightarrow{b} y \quad x \xrightarrow{c} z}{g(x) \xrightarrow{d} x}$$

the ntytt rule $\frac{x \xrightarrow{a} y}{g(f(x)) \xrightarrow{d} x}$ is irredundantly provable, but not with a linear proof. However, the ntytt rule $\frac{x \xrightarrow{a} y \quad x \xrightarrow{a} z}{g(f(x)) \xrightarrow{d} x}$ has a linear proof.

Clearly, each ntytt rule provable from a TSS is a substitution instance of an ntytt rule that has a linear proof. In Def. 7 it does not make any difference whether in clause 2 we quantify over rules $\frac{N}{V}$ that are provable (as in [13, 2, 8]), irredundantly provable (as in [9]), or linearly provable.

Lemma 2 If a rule $\frac{H}{\lambda}$ is linearly provable from a TSS P , then so is $\frac{\sigma(H)}{\sigma(\lambda)}$ for any substitution σ .

If a rule $\frac{H}{\lambda}$ as well as rules $\frac{H_\mu}{\mu}$ for each literal $\mu \in H$ are linearly provable, with the H_μ pairwise disjoint, then so is the rule $\frac{\bigcup_{\mu \in H} H_\mu}{\lambda}$.

Proof: Directly from the definition, and by composition of linear proofs. \square

In Sect. 2.6 a non-standard TSS P^+ is constructed out of a given TSS P in ready simulation format, via the intermediate stages P^\dagger and P^\ddagger . Here P^\ddagger consists of all xynft rules irredundantly provable from P^\dagger . With \hat{P}^\ddagger we denote the TSS consisting of all xynft rules linearly provable from P^\dagger . The TSS P^+ is obtained by augmenting P^\ddagger with non-standard rules; with \hat{P}^+ we denote the corresponding augmentation of \hat{P}^\ddagger . For the construction of the non-standard rules in the augmentation, it makes no difference whether we start from P^\ddagger or \hat{P}^\ddagger , since the difference disappears when abstracting from the right-hand sides of positive literals. Thus, $\hat{P}^+ \subseteq P^+$ and each rule in P^+ is a substitution instance of a rule in \hat{P}^+ . Hence, an ntytt rule is irredundantly provably from \hat{P}^+ iff it is irredundantly provable from P^+ .

Definition 14 A P -ruloid is called *linear* if it has a linear proof from \hat{P}^+ .

Clearly, each P -ruloid is a substitution instance of a linear P -ruloid. Consequently, Prop. 1 still holds if we only consider linear ruloids.

An essential part of our forthcoming congruence formats (cf. Def. 22) is a semantic requirement—“delay resistance”, Def. 19—on linear P -ruloids. To make the formats more easily applicable, we will show (in Thm. 8) that delay resistance is implied by a requirement on the rules of P . That result would fail if delay resistance were required for all P -ruloids. In the appendix it is indicated where linearity is used; see the remark after the proof of Lem. 11 as well as Ex. 8.

2.8 Decomposition of modal formulas

In [9] it was shown how one can decompose formulas from \mathbb{O} . To each term t and formula $\varphi \in \mathbb{O}$ a set $t^{-1}(\varphi)$ of decomposition mappings $\psi : V \rightarrow \mathbb{O}$ is assigned. Each of these mappings $\psi \in t^{-1}(\varphi)$ guarantees that for any closed substitution ρ , $\rho(t) \models \varphi$ if $\rho(x) \models \psi(x)$ for all $x \in \text{var}(t)$. Vice versa, whenever $\rho(t) \models \varphi$, there is a decomposition mapping $\psi \in t^{-1}(\varphi)$ with $\rho(x) \models \psi(x)$ for all $x \in \text{var}(t)$. This is formalised in Thm. 2.

Definition 15 [9] Let $P = (\Sigma, R)$ be a Γ -patient standard TSS in ready simulation format. We define $\cdot^{-1} : \mathbb{T}(\Sigma) \times \mathbb{O} \rightarrow \mathcal{P}(V \rightarrow \mathbb{O})$ as the function that for each $t \in \mathbb{T}(\Sigma)$ and $\varphi \in \mathbb{O}$ returns the set $t^{-1}(\varphi) \in \mathcal{P}(V \rightarrow \mathbb{O})$ of decomposition mappings $\psi : V \rightarrow \mathbb{O}$ generated by following five conditions. In the remainder of this definition, t denotes a univariate term, i.e. without multiple occurrences of the same variable.

1. $\psi \in t^{-1}(\bigwedge_{i \in I} \varphi_i)$ iff there are $\psi_i \in t^{-1}(\varphi_i)$ for each $i \in I$ such that

$$\psi(x) = \bigwedge_{i \in I} \psi_i(x) \quad \text{for all } x \in V$$

2. $\psi \in t^{-1}(\neg\varphi)$ iff there is a function $h : t^{-1}(\varphi) \rightarrow \text{var}(t)$ such that

$$\psi(x) = \begin{cases} \bigwedge_{\chi \in h^{-1}(x)} \neg\chi(x) & \text{if } x \in \text{var}(t) \\ \top & \text{if } x \notin \text{var}(t) \end{cases}$$

3. $\psi \in t^{-1}(\langle\alpha\rangle\varphi)$ iff there is a P -ruloid $\frac{H}{t \xrightarrow{\alpha} u}$ and a $\chi \in u^{-1}(\varphi)$ such that

$$\psi(x) = \begin{cases} \chi(x) \wedge \bigwedge_{x \xrightarrow{\beta} y \in H} \langle\beta\rangle\chi(y) \wedge \bigwedge_{x \xrightarrow{\gamma} \top \in H} \neg\langle\gamma\rangle\top & \text{if } x \in \text{var}(t) \\ \top & \text{if } x \notin \text{var}(t) \end{cases}$$

4. $\psi \in t^{-1}(\langle\varepsilon\rangle\varphi)$ iff one of the following holds:

- (a) either there is a $\chi \in t^{-1}(\varphi)$ such that

$$\psi(x) = \begin{cases} \langle\varepsilon\rangle\chi(x) & \text{if } x \text{ occurs } \Gamma\text{-liquid in } t \\ \chi(x) & \text{otherwise} \end{cases}$$

(b) or there is a Γ -impatient P -ruloid $\frac{H}{t \xrightarrow{\varepsilon} u}$ and a $\chi \in u^{-1}(\langle \varepsilon \rangle \varphi)$ such that

$$\psi(x) = \begin{cases} \langle \varepsilon \rangle \left(\chi(x) \wedge \bigwedge_{x \xrightarrow{\beta} y \in H} \langle \beta \rangle \chi(y) \wedge \bigwedge_{x \xrightarrow{\gamma} \top \in H} \neg \langle \gamma \rangle \top \right) & \text{if } x \text{ occurs} \\ & \Gamma\text{-liquid in } t \\ \chi(x) \wedge \bigwedge_{x \xrightarrow{\beta} y \in H} \langle \beta \rangle \chi(y) \wedge \bigwedge_{x \xrightarrow{\gamma} \top \in H} \neg \langle \gamma \rangle \top & \text{if } x \text{ occurs} \\ \top & \Gamma\text{-frozen in } t \\ & \text{if } x \notin \text{var}(t) \end{cases}$$

5. $\psi \in \sigma(t)^{-1}(\varphi)$ for a non-injective substitution $\sigma : \text{var}(t) \rightarrow V$ iff there is a $\chi \in t^{-1}(\varphi)$ such that

$$\psi(x) = \bigwedge_{z \in \sigma^{-1}(x)} \chi(z) \quad \text{for all } x \in V.$$

Lemma 3 Let $\psi \in t^{-1}(\varphi)$, for some term t and formula φ . If $x \notin \text{var}(t)$, then $\psi(x) \equiv \top$.

Theorem 2 [9] Let $P = (\Sigma, R)$ be a Γ -patient complete standard TSS in ready simulation format. For any term $t \in \mathbb{T}(\Sigma)$, closed substitution ρ , and $\varphi \in \mathbb{O}$:

$$\rho(t) \models \varphi \Leftrightarrow \exists \psi \in t^{-1}(\varphi) \forall x \in \text{var}(t) : \rho(x) \models \psi(x) .$$

3 Delay resistant TSSs

In the next section we will apply the decomposition method from Def. 15 and Thm. 2 to obtain congruence formats for (rooted) delay and weak bisimilarity. However, compared to branching and η -bisimilarity, which was the focus of [9], Def. 15 needs to be refined in the case $t^{-1}(\langle \varepsilon \rangle \varphi)$. This is because in the modal logics for delay and weak bisimilarity, occurrences of subformulas $\langle \beta \rangle \varphi'$ are always preceded by $\langle \varepsilon \rangle$, while in Def. 15 this is not always the case. The refinement of Def. 15, which is presented in Def. 20, is only valid for so-called delay resistant TSSs.

Def. 19 of delay resistance is inspired by a requirement in the RDB and RWB cool formats, see [14, Def. 15(3)]. It is crafted in such a way that Prop. 2 holds: if for a premise $x \xrightarrow{\beta} y$ in a ruloid $r = \frac{H}{t \xrightarrow{\alpha} u}$ the execution of β is delayed by a τ -step, i.e. for some substitution σ we merely have $\sigma(x) \xrightarrow{\tau} \xrightarrow{\beta} \sigma(y)$, we want that the conclusion $\sigma(t) \xrightarrow{\alpha} \sigma(u)$ of the σ -instance of r is unaffected, or merely delayed by a τ -step as well.

Example 3 Consider the rooted delay bisimilar processes p_0 and p_1 from the first LTS in Ex. 1. The ruloid r may apply when substituting p_0 for x and b for β , given that $p_0 \xrightarrow{b} 0$. If instead of p_0 we substitute p_1 for x , to safeguard the congruence property it is necessary that the (possibly delayed) conclusion of r can still be derived, even though we only have $p_1 \xrightarrow{\tau} q \xrightarrow{b} 0$.

In Def. 17 we allow two possible implementations of this idea. Each premise $x \xrightarrow{\beta} y$ of r is either *delayable* (Def. 16), in which case $\sigma(x) \xrightarrow{\tau} \xrightarrow{\beta} \sigma(y)$ induces $\sigma(t) \xrightarrow{\tau} \xrightarrow{\alpha} \sigma(u)$ by two ruloids that can be used in place of r ; or τ -pollable, meaning that r remains valid if this premise is replaced by $x \xrightarrow{\tau} z$ for a fresh variable z , so that the premise $\sigma(x) \xrightarrow{\tau} \sigma(z)$ takes over the role of $\sigma(x) \xrightarrow{\beta} \sigma(y)$.

Def. 17 and Prop. 2 allow only finitely many delayable positive premises, but infinitely many τ -pollable ones.

Definition 16 A premise $w \xrightarrow{\beta} y$ of an ntytt rule $r = \frac{H}{t \xrightarrow{\alpha} u}$ is *delayable* in a TSS P if there are ntytt rules $\frac{H_1}{t \xrightarrow{\tau} v}$ and $\frac{H_2}{v \xrightarrow{\alpha} u}$, linearly provable from P , with $H_1 \subseteq (H \setminus \{w \xrightarrow{\beta} y\}) \cup \{w \xrightarrow{\tau} z\}$ and $H_2 \subseteq (H \setminus \{w \xrightarrow{\beta} y\}) \cup \{z \xrightarrow{\beta} y\}$ for some term v and fresh variable z .

Suppose that $\frac{H}{t \xrightarrow{\alpha} u}$ in Def. 16 is a ruloid, so that w is a variable x . The intuition behind this definition is that the argument x of t may not be able to perform a β -transition to a term y immediately, but only after a τ -transition to z . The ruloid $\frac{H_1}{t \xrightarrow{\tau} v}$ then allows t to postpone its α -transition to u , by first performing a τ -transition to v . The ruloid $\frac{H_2}{v \xrightarrow{\alpha} u}$ guarantees that the postponed β -transition from z to y still gives rise to an α -transition from v to u .

Linearity is needed to make sure that in the construction of ruloids, distinct delayable positive premises are never collapsed to a single non-delayable premise.

Def. 17 requires that each positive premise is either delayable (i.e., in the finite set H^d), or τ -pollable, or redundant. Recall from Def. 5 the notation H^+ for the positive premises in H .

Definition 17 An ntytt rule $\frac{H}{t \xrightarrow{\alpha} u}$ is *positive delay resistant* w.r.t. a TSS P if there exists a finite set $H^d \subseteq H^+$ of delayable positive premises such that for each set $M \subseteq H^+ \setminus H^d$ there is a rule $r_M = \frac{H_M}{t \xrightarrow{\alpha} u}$, linearly provable from P , where $H_M \subseteq (H \setminus M) \cup M_\tau$ with $M_\tau = \{w \xrightarrow{\tau} z_y \mid (w \xrightarrow{\beta} y) \in M, z_y \text{ fresh}\}$.

The intuition behind Def. 18 is closely related to Def. 16. If a ruloid $\frac{H}{t \xrightarrow{\alpha} u}$ has a premise $x \xrightarrow{\gamma} \rightarrow$, we want it not to apply even if $\sigma(x)$ merely has a delayed γ -transition $\sigma(x) \xrightarrow{\tau} \xrightarrow{\gamma} \rightarrow$. We therefore require that for each premise $x \xrightarrow{\gamma} \rightarrow$ there must also be a premise $x \xrightarrow{\tau} \rightarrow$. However, we make an exception for redundant premises $x \xrightarrow{\gamma} \rightarrow$. Recall from Def. 5 the notation H^{s^-} for the stable negative premises in H .

Definition 18 A rule $\frac{H}{t \xrightarrow{\alpha} u}$ is *negative delay resistant* w.r.t. a TSS P if there is a rule $\frac{H'}{t \xrightarrow{\alpha} u}$, linearly provable from P , with $H' \subseteq H^+ \cup H^{s^-}$.

Definition 19 An ntytt rule $\frac{H}{t \xrightarrow{\alpha} u}$ is *delay resistant* w.r.t. a TSS P if it is positive delay resistant as well as negative delay resistant. A standard TSS P in ready simulation format is *delay resistant* if all its linear ruloids with a positive conclusion are delay resistant w.r.t. \hat{P}^+ .

The following proposition is key to the notion of delay resistance. It will allow us to adapt the definition of modal decomposition for delay resistant TSSs, so that it becomes applicable for generating congruence formats for weak semantics, like delay and weak bisimilarity, with a modal characterisation in which a modality $\langle \beta \rangle \varphi$ is always preceded by $\langle \varepsilon \rangle$.

Proposition 2 Let P be a delay resistant standard TSS in ready simulation format. Let $\frac{H}{t \xrightarrow{\alpha} u}$ be a P -ruloid and ρ a closed substitution such that $P \vdash_{ws} \rho(x) \xrightarrow{\varepsilon} \xrightarrow{\beta} \rho(y)$ for each premise $x \xrightarrow{\beta} y$ in H^+ and $P \vdash_{ws} \rho(x) \xrightarrow{\gamma} \rightarrow$ for each premise $x \xrightarrow{\gamma} \rightarrow$ in H^{s^-} . Then $P \vdash_{ws} \rho(t) \xrightarrow{\varepsilon} \xrightarrow{\alpha} \rho(u)$.

Proof: We prove the lemma for linear P -ruloids $\frac{H}{t \xrightarrow{\alpha} u}$; as each ruloid is a substitution instance of a linear ruloid, the result for general ruloids then follows.

We apply induction on the sum, over all premises $x \xrightarrow{\beta} y$ in H^+ , of the (smallest possible) number of τ -transitions in the sequence $\rho(x) \xrightarrow{\varepsilon} \rightarrow$. We note that the cases where this sum is infinite are in the proof immediately reduced to cases where this sum is finite.

Induction base: If the sum is zero, $P \vdash_{ws} \rho(x) \xrightarrow{\beta} \rho(y)$ for each $x \xrightarrow{\beta} y$ in H^+ . By Def. 18 there exists a P -ruloid $\frac{H'}{t \xrightarrow{\alpha} u}$ with $H' \subseteq H^+ \cup H^{s-}$. By assumption, $P \vdash_{ws} \rho(x) \xrightarrow{\gamma} \rho(y)$ for each $x \xrightarrow{\gamma} y$ in H' . So Prop. 1 yields $P \vdash_{ws} \rho(t) \xrightarrow{\alpha} \rho(u)$.

Induction step: Suppose the sum is positive. Take a finite set $H^d \subseteq H^+$ of delayable positive premises with the property ensured by Def. 17.

First we deal with the case that $P \not\vdash_{ws} \rho(x) \xrightarrow{\beta} \rho(y)$ for some $x \xrightarrow{\beta} y$ in $H^+ \setminus H^d$. Let M consist of those $x \xrightarrow{\beta} y$ in $H^+ \setminus H^d$ for which $P \vdash_{ws} \rho(x) \xrightarrow{\tau} q_y$ for some term q_y . Let ρ' be the closed substitution with $\rho'(z_y) = q_y$ for all right-hand sides y of such premises (where z_y is the right-hand side of the corresponding premise in M_τ), and ρ' coincides with ρ on all other variables. Then $\rho'(t) = \rho(t)$ and $\rho'(u) = \rho(u)$. Let $\frac{H_M}{t \xrightarrow{\alpha} u}$ be the linear P -ruloid that exists by Def. 17, with $H_M \subseteq (H \setminus M) \cup M_\tau$. Then $P \vdash_{ws} \rho'(x) \xrightarrow{\gamma} \rho'(y)$ for each $x \xrightarrow{\gamma} y$ in $H_M^{s-} \subseteq H^{s-}$. Moreover, we argue that $P \vdash_{ws} \rho'(x) \xrightarrow{\varepsilon} \xrightarrow{\beta} \rho'(y)$ for each premise $x \xrightarrow{\beta} y$ in H_M^+ . For each $x \xrightarrow{\beta} y$ in $H^+ \setminus M$ this is clear, because then $\rho'(x) = \rho(x)$ and $\rho'(y) = \rho(y)$. Furthermore, the definition of ρ' induces $P \vdash_{ws} \rho'(x) = \rho(x) \xrightarrow{\tau} \rho'(y)$ for each $x \xrightarrow{\tau} y$ in M_τ . We apply induction with regard to the P -ruloid $\frac{H_M}{t \xrightarrow{\alpha} u}$ and the closed substitution ρ' . Note that $P \vdash_{ws} \rho'(x) = \rho(x) \xrightarrow{\beta} \rho(y) = \rho'(y)$ for each $x \xrightarrow{\beta} y$ in $H^+ \setminus (H^d \cup M)$, because the definition of M induces $P \vdash_{ws} \rho(x) \xrightarrow{\tau} \rho(y)$. Also the premises $x \xrightarrow{\tau} y$ in M_τ do not contribute to the number of τ -transitions $\rho'(x) \xrightarrow{\varepsilon}$ on which we apply induction. Hence only premises $x \xrightarrow{\beta} y$ in H^d contribute to this number. As H^d is finite, this number is finite too. Since M is supposed to be non-empty, this number (for $\frac{H_M}{t \xrightarrow{\alpha} u}$ and ρ') is strictly smaller than for $\frac{H}{t \xrightarrow{\alpha} u}$ and ρ . An application of the induction hypothesis yields $P \vdash_{ws} \rho'(t) = \rho'(t) \xrightarrow{\alpha} \rho'(u) = \rho(u)$.

What remains is the case that $P \vdash_{ws} \rho(x) \xrightarrow{\beta} \rho(y)$ for all $x \xrightarrow{\beta} y$ in $H^+ \setminus H^d$. Then $P \not\vdash_{ws} \rho(x_0) \xrightarrow{\beta} \rho(y_0)$ for some $x_0 \xrightarrow{\beta} y_0$ in H^d . Let $P \vdash_{ws} \rho(x_0) \xrightarrow{\tau} p \xrightarrow{\varepsilon} \xrightarrow{\beta_0} \rho(y_0)$, where the closed term p is chosen so that $\xrightarrow{\varepsilon}$ is as short as possible. Let $H_0 = H \setminus \{x_0 \xrightarrow{\beta_0} y_0\}$. Since $x_0 \xrightarrow{\beta_0} y_0$ is a delayable premise of $\frac{H}{t \xrightarrow{\alpha} u}$, by Def. 16 there are linear P -ruloids $\frac{H_1}{t \xrightarrow{\tau} v}$ and $\frac{H_2}{v \xrightarrow{\alpha} u}$ with $H_1 \subseteq H_0 \cup \{x_0 \xrightarrow{\tau} z_0\}$ and $H_2 \subseteq H_0 \cup \{z_0 \xrightarrow{\beta_0} y_0\}$, for some term v and fresh variable z_0 . Let $\rho'(z_0) = p$ and ρ' coincides with ρ on all other variables. Since z_0 does not occur in H_0 , clearly $P \vdash_{ws} \rho'(x) \xrightarrow{\varepsilon} \xrightarrow{\beta} \rho'(y)$ for each $x \xrightarrow{\beta} y$ in H_0^+ and $P \vdash_{ws} \rho'(x) \xrightarrow{\gamma} \rho'(y)$ for each $x \xrightarrow{\gamma} y$ in H_0^{s-} . Moreover, $P \vdash_{ws} \rho'(x_0) \xrightarrow{\tau} \rho'(z_0)$. Compared with $\frac{H}{t \xrightarrow{\alpha} u}$ and ρ , in the case of $\frac{H_1}{t \xrightarrow{\tau} v}$ and ρ' the number of τ -transitions involved in the sequences $\xrightarrow{\varepsilon}$ has decreased. (As H^d is finite, these numbers are finite too.) So by induction, $P \vdash_{ws} \rho'(t) \xrightarrow{\varepsilon} \xrightarrow{\tau} \rho'(v)$. Furthermore, $P \vdash_{ws} \rho'(z_0) \xrightarrow{\varepsilon} \xrightarrow{\beta_0} \rho'(y_0)$. Again, compared with $\frac{H}{t \xrightarrow{\alpha} u}$ and ρ , in the case of $\frac{H_2}{v \xrightarrow{\alpha} u}$ and ρ' the number of τ -transitions involved in the sequences $\xrightarrow{\varepsilon}$ has decreased. So by induction, $P \vdash_{ws} \rho'(v) \xrightarrow{\varepsilon} \xrightarrow{\alpha} \rho'(u)$. Since $z_0 \notin \text{var}(t) \cup \text{var}(u)$, we conclude that $P \vdash_{ws} \rho(t) = \rho'(t) \xrightarrow{\varepsilon} \rho'(v) \xrightarrow{\varepsilon} \xrightarrow{\alpha} \rho'(u) = \rho(u)$. \square

As said before, for delay resistant TSSs case 4b of Def. 15, $t^{-1}(\langle \varepsilon \rangle \varphi)$, needs to be adapted, to ensure that in the modal logics for delay and weak bisimilarity, occurrences of subformulas $\langle \beta \rangle \varphi''$ are always preceded by $\langle \varepsilon \rangle$. Moreover, case 4a is provided with the restriction that φ is not of the form $\langle \alpha \rangle \varphi'$. Else decompositions of formulas $\langle \varepsilon \rangle \langle \alpha \rangle \varphi'$ in \mathbb{O}_d would be defined in terms of formulas $\chi \in t^{-1}(\langle \alpha \rangle \varphi')$, while $\langle \alpha \rangle \varphi'$ is not in \mathbb{O}_d . Instead, if φ is of the form $\langle \alpha \rangle \varphi'$, cases 3 and 4 of Def. 15 are combined, as can be seen in case 4b(ii) below.

Definition 20 Let P be a delay resistant Γ -patient standard TSS in ready simulation format. We define $\cdot_{\text{dr}}^{-1} : \mathbb{T}(\Sigma) \times \mathbb{O} \rightarrow \mathcal{P}(V \rightarrow \mathbb{O})$ exactly as \cdot^{-1} in Def. 15, except for case 4: $t_{\text{dr}}^{-1}(\langle \varepsilon \rangle \varphi)$ (with t univariate).

4. $\psi \in t_{\text{dr}}^{-1}(\langle \varepsilon \rangle \varphi)$ iff one of the following holds:

(a) either φ is not of the form $\langle \alpha \rangle \varphi'$, and there is a $\chi \in t_{\text{dr}}^{-1}(\varphi)$ such that

$$\psi(x) = \begin{cases} \langle \varepsilon \rangle \chi(x) & \text{if } x \text{ occurs } \Gamma\text{-liquid in } t \\ \chi(x) & \text{otherwise} \end{cases}$$

(b) or

(i) there is a Γ -impatient P -ruloid $\frac{H}{t \xrightarrow{\tau} u}$ and a $\chi \in u_{\text{dr}}^{-1}(\langle \varepsilon \rangle \varphi)$,

(ii) or φ is of the form $\langle \alpha \rangle \varphi'$, and there is a P -ruloid $\frac{H}{t \xrightarrow{\alpha} u}$ and a $\chi \in u_{\text{dr}}^{-1}(\varphi')$, such that

$$\psi(x) = \begin{cases} \langle \varepsilon \rangle \left(\chi(x) \wedge \bigwedge_{x \xrightarrow{\beta} y \in H^+} \langle \varepsilon \rangle \langle \beta \rangle \chi(y) \wedge \bigwedge_{x \xrightarrow{\gamma} \top \in H^{s-}} \neg \langle \gamma \rangle \top \right) & \text{if } x \text{ occurs} \\ & \Gamma\text{-liquid in } t \\ \chi(x) \wedge \bigwedge_{x \xrightarrow{\beta} y \in H^+} \langle \varepsilon \rangle \langle \beta \rangle \chi(y) \wedge \bigwedge_{x \xrightarrow{\gamma} \top \in H^{s-}} \neg \langle \gamma \rangle \top & \text{if } x \text{ occurs} \\ \top & \text{if } x \notin \text{var}(t). \end{cases}$$

The following theorem is the counterpart of Thm. 2 for delay resistant TSSs.

Theorem 3 Let $P = (\Sigma, R)$ be a delay resistant Γ -patient complete standard TSS in ready simulation format. For any $t \in \mathbb{T}(\Sigma)$, closed substitution ρ , and $\varphi \in \mathbb{O}$:

$$\rho(t) \models \varphi \Leftrightarrow \exists \psi \in t_{\text{dr}}^{-1}(\varphi) \forall x \in \text{var}(t) : \rho(x) \models \psi(x) .$$

Proof: By simultaneous induction on the structure of φ and the construction of ψ . We only treat the case where t is univariate; the case where t is not univariate is identical to the proof of Thm. 2 in [9]. The proof is by a case distinction on the syntactic structure of φ . We only treat the case $\varphi = \langle \varepsilon \rangle \varphi'$ here, because all other cases are identical to the proof of Thm. 2.

(\Rightarrow) Let $t_{\text{dr}}^{-1}(\langle \varepsilon \rangle \varphi')$ be defined as $t^{-1}(\langle \varepsilon \rangle \varphi')$ (case 4 of Def. 15), but using $\chi \in t_{\text{dr}}^{-1}(\varphi)$ instead of $\chi \in t^{-1}(\varphi)$ in case 4a, and $\chi \in u_{\text{dr}}^{-1}(\langle \varepsilon \rangle \varphi)$ instead of $\chi \in u^{-1}(\langle \varepsilon \rangle \varphi)$ in case 4b. By replaying the proof of Thm. 2 in [9], we find that $\rho(t) \models \langle \varepsilon \rangle \varphi' \Rightarrow \exists \psi \in t_{\text{dr}}^{-1}(\langle \varepsilon \rangle \varphi') \forall x \in \text{var}(t) : \rho(x) \models \psi(x)$. This in turn implies $\exists \psi \in t_{\text{dr}}^{-1}(\varphi) \forall x \in \text{var}(t) : \rho(x) \models \psi(x)$ using the fact that $\langle \beta \rangle \hat{\varphi}$ implies $\langle \varepsilon \rangle \langle \beta \rangle \hat{\varphi}$ for any modal formula $\hat{\varphi}$. Namely, in case φ' is not of the form $\langle \alpha \rangle \varphi''$, case 4 of Def. 15 and of Def. 20 only differ in dropping conjuncts $\neg \langle \gamma \rangle \top$ for negative premises not in H^{s-} , and in the addition of modalities $\langle \varepsilon \rangle$ in front of conjuncts $\langle \beta \rangle \chi(y)$, in case 4b, which make the definition of the formula $\psi(x)$ weaker. And although case 4a of Def. 15 is omitted from Def. 20 if $\varphi' = \langle \alpha \rangle \varphi''$, case 4b(ii) of Def. 20 guarantees that each $\psi \in t_{\text{dr}}^{-1}(\langle \varepsilon \rangle \langle \alpha \rangle \varphi'')$ is also in $t_{\text{dr}}^{-1}(\langle \varepsilon \rangle \langle \alpha \rangle \varphi'')$, but possibly with dropped conjuncts $\neg \langle \gamma \rangle \top$ and extra occurrences of $\langle \varepsilon \rangle$.

(\Leftarrow) Let $\psi \in t_{\text{dr}}^{-1}(\langle \varepsilon \rangle \varphi')$ with $\rho(x) \models \psi(x)$ for all $x \in \text{var}(t)$. The case where ψ is defined according to Def. 20.4a is identical to the treatment of this case in the proof of Thm. 2. We focus on the case where ψ is defined according to Def. 20.4b. Then there are either (case 4b(i)) a Γ -impatient P -ruloid $\frac{H}{t \xrightarrow{\tau} u}$ and

a $\chi \in u_{\text{dr}}^{-1}(\langle \varepsilon \rangle \varphi')$, or (case 4b(ii) with $\varphi' = \langle \alpha \rangle \varphi''$) a P -ruloid $\frac{H}{t \xrightarrow{\alpha} u}$ and a $\chi \in u_{\text{dr}}^{-1}(\varphi'')$, such that ψ is defined by

$$\psi(x) = \begin{cases} \langle \varepsilon \rangle \psi'(x) & \text{if } x \text{ occurs } \Gamma\text{-liquid in } t \\ \psi'(x) & \text{if } x \text{ occurs } \Gamma\text{-frozen in } t \\ \top & \text{if } x \notin \text{var}(t) \end{cases}$$

where

$$\psi'(x) = \chi(x) \wedge \bigwedge_{x \xrightarrow{\beta} y \in H^+} \langle \varepsilon \rangle \langle \beta \rangle \chi(y) \wedge \bigwedge_{x \xrightarrow{\gamma} \in H^{s-}} \neg \langle \gamma \rangle \top .$$

So for each x that occurs Γ -liquid in t , $\rho(x) \models \langle \varepsilon \rangle \psi'(x)$, i.e. for some p_x we have $P \vdash_{\text{ws}} \rho(x) \xrightarrow{\varepsilon} p_x \models \psi'(x)$. Moreover, $\rho(x) \models \psi'(x)$ for each x that occurs Γ -frozen in t . Define $\rho'(x) = p_x$ if x occurs Γ -liquid in t , and $\rho'(x) = \rho(x)$ otherwise. Since P is Γ -patient, $P \vdash_{\text{ws}} \rho(t) \xrightarrow{\varepsilon} \rho'(t)$. Furthermore, $\rho'(x) \models \psi'(x)$ for all $x \in \text{var}(t)$ implies: $\rho'(x) \models \chi(x)$ for all $x \in \text{var}(t)$; for each $x \xrightarrow{\beta} y \in H^+$ we have $\rho'(x) \models \langle \varepsilon \rangle \langle \beta \rangle \chi(y)$, i.e. $P \vdash_{\text{ws}} \rho'(x) \xrightarrow{\varepsilon} \xrightarrow{\beta} p_y \models \chi(y)$ for some p_y ; and for each $x \xrightarrow{\gamma} \in H^{s-}$ we have $P \not\vdash_{\text{ws}} \rho'(x) \xrightarrow{\gamma} q$ for all q , so $P \vdash_{\text{ws}} \rho'(x) \not\xrightarrow{\gamma}$ by the completeness of P . Define $\rho''(x) = \rho'(x)$ and $\rho''(y) = p_y$ for all $x \in \text{var}(t)$ and $x \xrightarrow{\beta} y \in H$. Then $\rho''(z) \models \chi(z)$ for all $z \in \text{var}(u)$. Moreover, $P \vdash_{\text{ws}} \rho''(x) \xrightarrow{\varepsilon} \xrightarrow{\beta} \rho''(y)$ for each premise $x \xrightarrow{\beta} y$ in H^+ , whereas $P \vdash_{\text{ws}} \rho''(x) \not\xrightarrow{\gamma}$ for each premise $x \xrightarrow{\gamma} \in H^{s-}$.

CASE 4b(i): Prop. 2 yields $P \vdash_{\text{ws}} \rho''(t) \xrightarrow{\varepsilon} \xrightarrow{\tau} \rho''(u)$. By induction on the construction of ψ , $\rho''(u) \models \langle \varepsilon \rangle \varphi'$. Since $P \vdash_{\text{ws}} \rho(t) \xrightarrow{\varepsilon} \rho'(t)$ and $\rho'(t) = \rho''(t)$, it follows that $\rho(t) \models \langle \varepsilon \rangle \varphi'$.

CASE 4b(ii): Prop. 2 yields $P \vdash_{\text{ws}} \rho''(t) \xrightarrow{\varepsilon} \xrightarrow{\alpha} \rho''(u)$. By induction on formula size, $\rho''(u) \models \varphi''$. Since $P \vdash_{\text{ws}} \rho(t) \xrightarrow{\varepsilon} \rho'(t)$ and $\rho'(t) = \rho''(t)$, it follows that $\rho(t) \models \langle \varepsilon \rangle \langle \alpha \rangle \varphi''$. \square

4 Rooted delay and weak bisimilarity as a congruence

A behavioural equivalence \sim is a *congruence* for a function symbol f defined on an LTS if $p_i \sim q_i$ for all $i \in \{1, \dots, \text{ar}(f)\}$ implies that $f(p_1, \dots, p_{\text{ar}(f)}) \sim f(q_1, \dots, q_{\text{ar}(f)})$. We call \sim a congruence for a TSS (Σ, R) , if it is a congruence for all function symbols from the signature Σ with respect to the LTS generated by (Σ, R) . This is the case if for any open term $t \in \mathbb{T}(\Sigma)$ and any closed substitutions $\rho, \rho' : V \rightarrow \mathbb{T}$ we have that

$$\forall x \in \text{var}(t) : \rho(x) \sim \rho'(x) \Rightarrow \rho(t) \sim \rho'(t) .$$

A *congruence format* for \sim is a list of syntactic restrictions on TSSs, such that \sim is guaranteed to be a congruence for any TSS satisfying these restrictions.

We proceed to apply the decomposition method from the previous section to derive congruence formats for delay bisimulation and rooted delay bisimulation semantics. The idea behind the construction of these congruence formats is that it must be guaranteed that a formula from the characterising logic of the equivalence under consideration is always decomposed into formulas from this same logic. We prove that the delay bisimulation format guarantees that a formula from \mathbb{O}_d is always decomposed into formulas from \mathbb{O}_d^{\equiv} (see Prop. 4). Likewise, the rooted delay bisimulation format guarantees that a formula from \mathbb{O}_{rd} is always decomposed into formulas from \mathbb{O}_{rd}^{\equiv} (see Prop. 5). This implies the desired congruence results (see Thm. 4 and Thm. 5, respectively).

At the end of this section it is sketched how these results can be transposed to (rooted) weak bisimilarity, by adding one condition to the congruence format for (rooted) delay bisimilarity.

4.1 Congruence format for rooted delay bisimilarity

We recall the notion of a rooted branching bisimulation safe rule, which underlies the rooted branching bisimulation format from [9]. The congruence format for rooted delay bisimilarity is obtained by additionally requiring delay resistance.

We assume two predicates on arguments of function symbols from [5, 9]. The predicate Λ marks arguments that contain processes that have started executing (but may currently be unable to execute). The predicate \aleph marks arguments that contain processes that can execute immediately. For example, in process algebra, Λ and \aleph hold for the arguments of the merge $t_1 \parallel t_2$, and for the first argument of sequential composition $t_1 \cdot t_2$; they can contain processes that started to execute in the past, and these processes can continue their execution immediately. On the other hand, Λ and \aleph typically do not hold for the second argument of sequential composition; it contains a process that did not yet start to execute, and cannot execute immediately (in absence of the empty process). Λ does not hold and \aleph holds for the arguments of alternative composition $t_1 + t_2$; they contain processes that did not yet start to execute, but that can start executing immediately.

Definition 21 [9] A standard ntytt rule $r = \frac{H}{t \xrightarrow{\alpha} u}$ is *rooted branching bisimulation safe* w.r.t. \aleph and Λ if it satisfies the following conditions. Let $x \in \text{var}(t)$.

1. Right-hand sides of positive premises occur only Λ -liquid in u .
2. If x occurs only Λ -liquid in t , then x occurs only Λ -liquid in r .
3. If x occurs only \aleph -frozen in t , then x occurs only \aleph -frozen in H .
4. If x has exactly one \aleph -liquid occurrence in t , which is also Λ -liquid, then x has at most one \aleph -liquid occurrence in H , which must be in a positive premise. If moreover this premise is labelled τ , then r must be $\aleph \cap \Lambda$ -patient.

Definition 22 A standard TSS P is in *rooted delay bisimulation format* if it is in ready simulation format and delay resistant, and, for some \aleph and Λ , it is $\aleph \cap \Lambda$ -patient and all its transition rules are rooted branching bisimulation safe w.r.t. \aleph and Λ .

This TSS is in *delay bisimulation format* if moreover Λ is universal.

Remark 1 If a standard TSS P is in rooted delay bisimulation format, then there are smallest predicates \aleph_0 and Λ_0 such that P is in rooted delay bisimulation format w.r.t. \aleph_0 and Λ_0 . Namely the Λ_0 -liquid arguments are *generated* by conditions 1 and 2 of Def. 21; they are the smallest collection of arguments such that these two requirements are satisfied. Likewise the \aleph_0 -liquid arguments are generated by condition 3, which can be read as “If x occurs \aleph -liquid in H , then the unique occurrence of x in t is \aleph -liquid.” For any standard TSS P in ready simulation format, \aleph_0 and Λ_0 are determined in this way, and whether P is in rooted delay bisimulation format then depends solely on whether it is delay resistant and $\aleph_0 \cap \Lambda_0$ -patient, and condition 4 of Def. 21 is satisfied by all rules in P .

4.2 Preservation of syntactic restrictions

In the definition of modal decomposition, we did not use the rules from the original delay resistant standard TSS P , but the P -ruloids. Therefore we must verify that if P is in rooted delay bisimulation format, then the P -ruloids are rooted delay bisimulation safe (Prop. 3). In the proof of this preservation

result, rules with a negative conclusion play an important role. For this reason, the notion of rooted branching bisimulation safety is extended to non-standard rules.

Definition 23 [9] An ntytt rule $r = \frac{H}{t \xrightarrow{\varphi} \tau}$ is *rooted branching bisimulation safe* w.r.t. \aleph and Λ if it satisfies conditions 2 and 3 of Def. 21.

Proposition 3 [9] Let P be a standard TSS in ready simulation format, in which each transition rule is rooted branching bisimulation safe w.r.t. \aleph and Λ . Then each P -ruloid is rooted branching bisimulation safe w.r.t. \aleph and Λ .

The following lemma is a crucial step in the proof of Prop. 3.

Lemma 4 [9] Let P be a TSS in decent ntyft format, in which each transition rule is rooted branching bisimulation safe w.r.t. \aleph and Λ . Then any ntytt rule irredundantly provable from P is rooted branching bisimulation safe w.r.t. \aleph and Λ .

4.3 Preservation of modal characterisations

Consider a standard TSS that is in rooted delay bisimulation format, w.r.t. some \aleph and Λ . Def. 20 yields decomposition mappings $\psi \in t_{\text{dr}}^{-1}(\varphi)$, with $\Gamma = \aleph \cap \Lambda$. In this section we will first prove that if $\varphi \in \mathbb{O}_d$, then $\psi(x) \in \mathbb{O}_d^{\equiv}$ if x occurs only Λ -liquid in t . (That is why in the delay bisimulation format, Λ must be universal.) Next we will prove that if $\varphi \in \mathbb{O}_{rd}$, then $\psi(x) \in \mathbb{O}_{rd}^{\equiv}$ for all variables x . From these preservation results we will, in Sect. 4.4, deduce the promised congruence results for delay bisimilarity and rooted delay bisimilarity, respectively.

Proposition 4 Let P be a delay resistant, $\aleph \cap \Lambda$ -patient standard TSS in ready simulation format, in which each transition rule is rooted branching bisimulation safe w.r.t. \aleph and Λ . For any term t and variable x that occurs only Λ -liquid in t :

$$\varphi \in \mathbb{O}_d \Rightarrow \forall \psi \in t_{\text{dr}}^{-1}(\varphi) : \psi(x) \in \mathbb{O}_d^{\equiv} .$$

Proof: We apply simultaneous induction on the structure of $\varphi \in \mathbb{O}_d$ and the construction of ψ . Let $\psi \in t_{\text{dr}}^{-1}(\varphi)$, and let x occur only Λ -liquid in t . First we treat the case where t is univariate. If $x \notin \text{var}(t)$, then by Lem. 3, $\psi(x) \equiv \top \in \mathbb{O}_d^{\equiv}$. So suppose x has exactly one, Λ -liquid occurrence in t .

We need to consider the four possible syntactic forms of φ in the BNF grammar of \mathbb{O}_d in Def. 4.

- $\varphi = \bigwedge_{i \in I} \varphi_i$ with $\varphi_i \in \mathbb{O}_d$ for each $i \in I$. By Def. 15.1, $\psi(x) = \bigwedge_{i \in I} \psi_i(x)$ with $\psi_i \in t_{\text{dr}}^{-1}(\varphi_i)$ for each $i \in I$. By induction on formula size, $\psi_i(x) \in \mathbb{O}_d^{\equiv}$ for each $i \in I$, so $\psi(x) \in \mathbb{O}_d^{\equiv}$.
- $\varphi = \neg \varphi'$ with $\varphi' \in \mathbb{O}_d$. By Def. 15.2, there is a function $h : t_{\text{dr}}^{-1}(\varphi') \rightarrow \text{var}(t)$ such that $\psi(x) = \bigwedge_{\chi \in h^{-1}(x)} \neg \chi(x)$. By induction on formula size, $\chi(x) \in \mathbb{O}_d^{\equiv}$ for each $\chi \in h^{-1}(x)$, so $\psi(x) \in \mathbb{O}_d^{\equiv}$.
- $\varphi = \langle \varepsilon \rangle \varphi'$ with $\varphi' \in \mathbb{O}_d$ (which implies that φ' is not of the form $\langle a \rangle \varphi''$). According to Def. 20.4, we can distinguish two cases.

CASE 1: $\psi(x)$ is defined on the basis of Def. 20.4a. Then either $\psi(x) = \langle \varepsilon \rangle \chi(x)$ or $\psi(x) = \chi(x)$ for some $\chi \in t_{\text{dr}}^{-1}(\varphi')$. By induction on formula size, $\chi(x) \in \mathbb{O}_d^{\equiv}$. So $\psi(x) \in \mathbb{O}_d^{\equiv}$.

CASE 2: $\psi(x)$ is defined on the basis of Def. 20.4b, employing an $\aleph \cap \Lambda$ -impatient P -ruloid $\frac{H}{t \xrightarrow{\tau} u}$ and a $\chi \in u_{\text{dr}}^{-1}(\langle \varepsilon \rangle \varphi')$. The rest of this case proceeds exactly as the next case below.

- $\varphi = \langle \varepsilon \rangle \langle a \rangle \varphi'$ with $\varphi' \in \mathbb{O}_d$. Then $\psi(x)$ is defined on the basis of Def. 20.4(b), employing either an $\aleph \cap \Lambda$ -impatient P -ruloid $\frac{H}{t \xrightarrow{\tau} u}$ and a $\chi \in u_{\text{dr}}^{-1}(\langle \varepsilon \rangle \langle a \rangle \varphi')$, or a P -ruloid $\frac{H}{t \xrightarrow{a} u}$ and a $\chi \in u_{\text{dr}}^{-1}(\varphi')$. By Prop. 3 we can assume that $\frac{H}{t \xrightarrow{\tau} u}$ or $\frac{H}{t \xrightarrow{a} u}$ is rooted branching bisimulation safe w.r.t. \aleph and Λ . Since the occurrence of x in t is Λ -liquid, by condition 2 of Def. 21, x occurs only Λ -liquid in u . Moreover, by condition 1 of Def. 21, variables in $\text{rhs}(H)$ occur only Λ -liquid in u . So by induction on the construction of ψ or on formula size, $\chi(x) \in \mathbb{O}_d^{\equiv}$, and $\chi(y) \in \mathbb{O}_d^{\equiv}$ for each $x \xrightarrow{\beta} y \in H$. We distinguish two cases.

CASE 1: The occurrence of x in t is \aleph -frozen. By condition 3 of Def. 21, x does not occur in H . Hence $\psi(x) = \chi(x) \in \mathbb{O}_d^{\equiv}$.

CASE 2: The occurrence of x in t is \aleph -liquid. Since moreover the occurrence of x in t is Λ -liquid, by condition 4 of Def. 21, x is the left-hand side of at most one premise in H , which is positive. And since $\frac{H}{t \xrightarrow{\tau} u}$ or $\frac{H}{t \xrightarrow{a} u}$ is $\aleph \cap \Lambda$ -impatient, this positive premise does not carry the label τ . Hence $\psi(x) = \langle \varepsilon \rangle \chi(x)$ or $\psi(x) = \langle \varepsilon \rangle (\chi(x) \wedge \langle \varepsilon \rangle \langle b \rangle \chi(y))$ with $b \in A$ and $x \xrightarrow{b} y \in H$. In either case, $\psi(x) \in \mathbb{O}_d^{\equiv}$.

Finally, we treat the case where t is not univariate. Then $t = \sigma(u)$ for some univariate term u and non-injective mapping $\sigma : \text{var}(u) \rightarrow V$. By Def. 15.5, $\psi(x) = \bigwedge_{z \in \sigma^{-1}(x)} \chi(z)$ for some $\chi \in u_{\text{dr}}^{-1}(\varphi)$. Since u is univariate, and for each $z \in \sigma^{-1}(x)$ the occurrence in u is Λ -liquid, $\chi(z) \in \mathbb{O}_d^{\equiv}$ for all $z \in \sigma^{-1}(x)$. Hence $\psi(x) \in \mathbb{O}_d^{\equiv}$. \square

Proposition 5 Let P be a delay resistant, $\aleph \cap \Lambda$ -patient standard TSS in ready simulation format, in which each transition rule is branching delay bisimulation safe w.r.t. \aleph and Λ . For any term t and variable x :

$$\varphi \in \mathbb{O}_{rd} \Rightarrow \forall \psi \in t_{\text{dr}}^{-1}(\varphi) : \psi(x) \in \mathbb{O}_{rd}^{\equiv} .$$

Proof: We apply simultaneous induction on the structure of $\varphi \in \mathbb{O}_{rd}$ and the construction of ψ . Let $\psi \in t_{\text{dr}}^{-1}(\varphi)$. We restrict attention to the case where t is univariate; the general case then follows just as at the end of the proof of Prop. 4. If $x \notin \text{var}(t)$, then by Lem. 3, $\psi(x) \equiv \top \in \mathbb{O}_{rd}^{\equiv}$. So suppose x occurs once in t .

- The cases $\varphi = \bigwedge_{i \in I} \varphi_i$ and $\varphi = \neg \varphi'$ proceed as in the proof of Prop. 4, replacing \mathbb{O}_d by \mathbb{O}_{rd} .
- $\varphi = \langle \varepsilon \rangle \langle \alpha \rangle \varphi'$ with $\varphi' \in \mathbb{O}_d$. According to Def. 20.4(b),

$$\psi(x) = \begin{cases} \langle \varepsilon \rangle \left(\chi(x) \wedge \bigwedge_{x \xrightarrow{\beta} y \in H^+} \langle \varepsilon \rangle \langle \beta \rangle \chi(y) \wedge \bigwedge_{x \xrightarrow{\gamma} \top \in H^{s^-}} \neg \langle \gamma \rangle \top \right) & \text{if } x \text{ occurs} \\ & \aleph \cap \Lambda\text{-liquid in } t \\ \chi(x) \wedge \bigwedge_{x \xrightarrow{\beta} y \in H^+} \langle \varepsilon \rangle \langle \beta \rangle \chi(y) \wedge \bigwedge_{x \xrightarrow{\gamma} \top \in H^{s^-}} \neg \langle \gamma \rangle \top & \text{if } x \text{ occurs} \\ & \aleph \cap \Lambda\text{-frozen in } t \end{cases}$$

where either there is an $\aleph \cap \Lambda$ -impatient P -ruloid $\frac{H}{t \xrightarrow{\tau} u}$ with $\chi \in u_{\text{dr}}^{-1}(\langle \varepsilon \rangle \langle \alpha \rangle \varphi')$, or there is a P -ruloid $\frac{H}{t \xrightarrow{a} u}$ with $\chi \in u_{\text{dr}}^{-1}(\varphi')$.

If H^{s^-} contains a negative premise with left-hand side x , then by the definition of H^{s^-} , it also contains $x \xrightarrow{\tau} \top$. Clearly $p \models \neg \langle \tau \rangle \top \wedge \neg \langle \gamma \rangle \top$ if and only if $p \models \neg \langle \tau \rangle \top \wedge \neg \langle \varepsilon \rangle \langle \gamma \rangle \top$, and moreover

$p \models \neg\langle\tau\rangle\top$ if and only if $p \models \neg\langle\varepsilon\rangle\langle\tau\rangle\top$, for any process p and action γ . This implies that the conjuncts $\bigwedge_{x \xrightarrow{\gamma} y \in H^{s-}} \neg\langle\gamma\rangle\top$ in $\psi(x)$ can be replaced by

$$\bigwedge_{x \xrightarrow{\gamma} y \in H^{s-}} \neg\langle\varepsilon\rangle\langle\gamma\rangle\top .$$

By induction on the construction of ψ or on formula size, $\chi(x) \in \mathbb{O}_{rd}^{\equiv}$. By Prop. 3 we can assume that $\frac{H}{t \xrightarrow{\tau} u}$ or $\frac{H}{t \xrightarrow{\alpha} u}$ is rooted branching bisimulation safe w.r.t. \aleph and Λ ; so by condition 1 of Def. 21, variables in $rhs(H)$ occur only Λ -liquid in u . Hence, by Prop. 4, $\chi(y) \in \mathbb{O}_d^{\equiv}$ for each $x \xrightarrow{\beta} y \in H$. In case the occurrence of x in t is $\aleph \cap \Lambda$ -frozen, this immediately yields $\psi(x) \in \mathbb{O}_{rd}^{\equiv}$.

So suppose the occurrence of x in t is $\aleph \cap \Lambda$ -liquid. By condition 2 of Def. 21, x occurs only Λ -liquid in u , so by Prop. 4, $\chi(x) \in \mathbb{O}_d^{\equiv}$. And by condition 4 of Def. 21, x is the left-hand side of at most one premise in H , which is positive. So either $\psi(x) = \langle\varepsilon\rangle\chi(x)$, or $\psi(x) = \langle\varepsilon\rangle(\chi(x) \wedge \langle\varepsilon\rangle\langle\beta\rangle\chi(y))$ with $x \xrightarrow{\beta} y \in H$. In the first case, and in the second case with $\beta \neq \tau$, this immediately yields $\psi(x) \in \mathbb{O}_d^{\equiv} \subset \mathbb{O}_{rd}^{\equiv}$.

Consider the second case with $\beta = \tau$. By condition 4 of Def. 21, $\frac{H}{t \xrightarrow{\alpha} u}$ or $\frac{H}{t \xrightarrow{\tau} u}$ is $\aleph \cap \Lambda$ -patient. So clearly this P -ruloid it is of the form $\frac{x \xrightarrow{\tau} y}{C[x] \xrightarrow{\tau} C[y]}$. Since t is univariate, it follows that $x \notin var(u)$. Hence, by Lem. 3, $\chi(x) \equiv \top$. Thus $\psi(x) \equiv \langle\varepsilon\rangle\langle\beta\rangle\chi(y) \in \mathbb{O}_{rd}^{\equiv}$.

- $\varphi \in \mathbb{O}_d$. The cases $\varphi = \bigwedge_{i \in I} \varphi_i$ and $\varphi = \neg\varphi'$ proceed as in the proof of Prop. 4, replacing \mathbb{O}_d^{\equiv} by \mathbb{O}_{rd}^{\equiv} , and the case $\varphi = \langle\varepsilon\rangle\langle a \rangle\varphi'$ was already treated above. The remaining case is $\varphi = \langle\varepsilon\rangle\varphi''$ with φ'' not of the form $\langle a \rangle\varphi'$. If the occurrence of x in t is Λ -liquid, then by Prop. 4, $\psi(x) \in \mathbb{O}_d^{\equiv} \subset \mathbb{O}_{rd}^{\equiv}$. So we can assume that this occurrence is Λ -frozen. According to Def. 20.4 we can distinguish two cases.

CASE 1: $\psi(x)$ is defined on the basis of case 4a. Then $\psi(x) = \chi(x)$ for some $\chi \in t_{dr}^{-1}(\varphi'')$. By induction on formula size, $\psi(x) = \chi(x) \in \mathbb{O}_{rd}^{\equiv}$.

CASE 2: $\psi(x)$ is defined on the basis of case 4b, employing an $\aleph \cap \Lambda$ -impatient P -ruloid $\frac{H}{t \xrightarrow{\tau} u}$ and a $\chi \in u_{dr}^{-1}(\langle\varepsilon\rangle\varphi'')$. Then

$$\psi(x) = \chi(x) \wedge \bigwedge_{x \xrightarrow{\beta} y \in H^+} \langle\varepsilon\rangle\langle\beta\rangle\chi(y) \wedge \bigwedge_{x \xrightarrow{\gamma} y \in H^{s-}} \neg\langle\gamma\rangle\top .$$

By induction on the construction of ψ , $\chi(x) \in \mathbb{O}_{rd}^{\equiv}$. By Prop. 3 we can assume that $\frac{H}{t \xrightarrow{\tau} u}$ is rooted branching bisimulation safe w.r.t. \aleph and Λ ; so by condition 1 of Def. 21, variables in $rhs(H)$ occur only Λ -liquid in u . Hence, by Prop. 4, $\chi(y) \in \mathbb{O}_d^{\equiv}$ for each $x \xrightarrow{\beta} y \in H$. And similar as in the previous case $\varphi = \langle\varepsilon\rangle\langle\alpha\rangle\varphi'$ with $\varphi' \in \mathbb{O}_d$ we can argue that the conjuncts $\neg\langle\gamma\rangle\top$ in $\psi(x)$ can be replaced by $\neg\langle\varepsilon\rangle\langle\gamma\rangle\top$. Hence $\psi(x) \in \mathbb{O}_{rd}^{\equiv}$. \square

4.4 Congruence for rooted delay bisimilarity

Now we are in a position to prove the promised congruence results for \leftrightarrow_d and \leftrightarrow_{rd} .

Theorem 4 Let P be a complete standard TSS in delay bisimulation format. Then \leftrightarrow_d is a congruence for P .

Proof: By Def. 22 each rule of P is rooted branching bisimulation safe w.r.t some \aleph and the universal predicate Λ , and P is delay resistant, $\aleph \cap \Lambda$ -patient and in ready simulation format.

Let ρ, ρ' be closed substitutions and t a term. Suppose that $\rho(x) \dot{\leftrightarrow}_d \rho'(x)$ for all $x \in \text{var}(t)$; we need to prove that then $\rho(t) \dot{\leftrightarrow}_d \rho'(t)$.

Let $\rho(t) \models \varphi \in \mathbb{O}_d$. By Thm. 3, taking $\Gamma = \aleph \cap \Lambda$, there is a $\psi \in t_{\text{dr}}^{-1}(\varphi)$ with $\rho(x) \models \psi(x)$ for all $x \in \text{var}(t)$. Since x occurs Λ -liquid in t (because Λ is universal), by Prop. 4, $\psi(x) \in \mathbb{O}_d^{\equiv}$ for all $x \in \text{var}(t)$. By Thm. 1, $\rho(x) \dot{\leftrightarrow}_d \rho'(x)$ implies $\rho(x) \sim_{\mathbb{O}_d^{\equiv}} \rho'(x)$ for all $x \in \text{var}(t)$. So $\rho'(x) \models \psi(x)$ for all $x \in \text{var}(t)$. Therefore, by Thm. 3, $\rho'(t) \models \varphi$. Likewise, $\rho'(t) \models \varphi \in \mathbb{O}_d$ implies $\rho(t) \models \varphi$. So $\rho(t) \sim_{\mathbb{O}_d} \rho'(t)$. Hence, by Thm. 1, $\rho(t) \dot{\leftrightarrow}_d \rho'(t)$. \square

We can follow the same approach to prove that the rooted delay bisimulation format guarantees that $\dot{\leftrightarrow}_{rd}$ is a congruence.

Theorem 5 Let P be a complete standard TSS in rooted delay bisimulation format. Then $\dot{\leftrightarrow}_{rd}$ is a congruence for P .

The proof of Thm. 5 is similar to the one of Thm. 4, except that Prop. 5 is applied instead of Prop. 4; therefore x need not occur Λ -liquid in t , which is why universality of Λ can be dropped.

4.5 Rooted weak bisimilarity as a congruence

We now proceed to derive a congruence format for rooted weak bisimilarity. It is obtained from the congruence format from [9] for rooted η -bisimilarity by additionally requiring delay resistance. The format for rooted η -bisimilarity in turn is obtained by strengthening condition 1 in the definition of rooted branching bisimulation safeness.

Definition 24 [9] An ntytt rule $r = \frac{H}{t \xrightarrow{\alpha} u}$ is *rooted η -bisimulation safe* w.r.t. \aleph and Λ if it satisfies conditions 2–4 of Def. 21, together with:

- 1'. Right-hand sides of positive premises occur only $\aleph \cap \Lambda$ -liquid in u .

Definition 25 A standard TSS is in *rooted weak bisimulation format* if it is in ready simulation format and delay resistant, and, for some \aleph and Λ , it is $\aleph \cap \Lambda$ -patient and contains only rules that are rooted η -bisimulation safe w.r.t. \aleph and Λ .

This TSS is in *weak bisimulation format* if moreover Λ is universal.

The proofs that these formats guarantee that [rooted] weak bisimilarity is a congruence are largely identical to the proofs for the [rooted] delay bisimulation format. We will therefore only explain where these proofs differ.

For non-standard ntytt rules, the notion of rooted η -bisimulation safeness coincides with the notion of rooted branching bisimulation safeness (see Def. 23).

Proposition 6 [9] Let P be a TSS in ready simulation format, in which each transition rule is rooted η -bisimulation safe w.r.t. \aleph and Λ . Then each P -ruloid is rooted η -bisimulation safe w.r.t. \aleph and Λ .

To adapt Prop. 4 from rooted delay to rooted weak bisimulation safeness, the following lemma is needed, with regard to Def. 20.

Lemma 5 Given a Γ -patient standard TSS $P = (\Sigma, R)$ in ready simulation format. Let $\psi \in t_{\text{dr}}^{-1}(\langle \varepsilon \rangle \varphi)$ for some term t and formula φ . If x occurs only Γ -liquid in t , then $\psi(x) \equiv \langle \varepsilon \rangle \psi(x)$.

Proof: Let x occur only Γ -liquid in t . In case t is univariate, it follows immediately from Def. 20 that $\psi(x) \equiv \langle \varepsilon \rangle \varphi'$ for some formula φ' . So for general terms t , $\psi(x) \equiv \bigwedge_{i \in I} \langle \varepsilon \rangle \varphi_i$. This implies $\psi(x) \equiv \langle \varepsilon \rangle \psi(x)$. \square

The proof of the following proposition is very similar to the proof of the corresponding Prop. 4 for the rooted delay bisimulation format.

Proposition 7 Let P be a delay resistant, $\aleph \cap \Lambda$ -patient standard TSS in ready simulation format, in which each transition rule is rooted η -bisimulation safe w.r.t. \aleph and Λ . For any term t and variable x that occurs only Λ -liquid in t :

$$\varphi \in \mathbb{O}_w \Rightarrow \forall \psi \in t_{\text{dr}}^{-1}(\varphi) : \psi(x) \in \mathbb{O}_w^{\equiv} .$$

In the case $\varphi = \langle \varepsilon \rangle \langle a \rangle \langle \varepsilon \rangle \varphi'$ with $\varphi' \in \mathbb{O}_w$ the proof of Prop. 7 slightly deviates from the case $\varphi = \langle \varepsilon \rangle \langle a \rangle \varphi'$ with $\varphi' \in \mathbb{O}_d$ in the proof of Prop. 4. At the end of the latter case, in CASE 2 where the occurrence of x in univariate term t is \aleph -liquid, it may be that $\psi(x) = \langle \varepsilon \rangle (\chi(x) \wedge \langle \varepsilon \rangle \langle b \rangle \chi(y))$ with $x \xrightarrow{b} y \in H$. The additional observation we make in the proof of Prop. 7 is that owing to the stronger condition 1', y only occurs $\aleph \cap \Lambda$ -liquid in u . So according to Lem. 5 with $\Gamma = \aleph \cap \Lambda$, $\chi(y) \equiv \langle \varepsilon \rangle \chi(y)$. Hence, $\psi(x) \equiv \langle \varepsilon \rangle (\chi(x) \wedge \langle \varepsilon \rangle \langle b \rangle \langle \varepsilon \rangle \chi(y)) \in \mathbb{O}_w^{\equiv}$.

The same difference with the proof of Prop. 4 appears in the case $\varphi = \langle \varepsilon \rangle \varphi'$ with $\varphi' \in \mathbb{O}_w$, CASE 2. For the rest, the proofs proceed in exactly the same way.

The proof of the following proposition is very similar to the proof of the corresponding Prop. 5 for the rooted delay bisimulation format.

Proposition 8 Let P be a delay resistant, $\aleph \cap \Lambda$ -patient standard TSS in ready simulation format, in which each transition rule is rooted η -bisimulation safe w.r.t. \aleph and Λ . For any term t and variable x :

$$\varphi \in \mathbb{O}_{rw} \Rightarrow \forall \psi \in t_{\text{dr}}^{-1}(\varphi) : \psi(x) \in \mathbb{O}_{rw}^{\equiv} .$$

Again the only real difference with the proof of Prop. 5 is that we need to exploit the stronger condition 1' of Def. 24: for each P -ruloid $\frac{H}{t \xrightarrow{\alpha} u}$, each $y \in \text{rhs}(H)$ can occur only $\aleph \cap \Lambda$ -liquid in u ; so by Lem. 5 with $\Gamma = \aleph \cap \Lambda$, $\chi \in u_{\text{dr}}^{-1}(\langle \varepsilon \rangle \varphi)$ implies $\chi(y) \equiv \langle \varepsilon \rangle \chi(y)$. Moreover, we need to observe that $\neg \langle \varepsilon \rangle \langle \gamma \rangle \top \equiv \neg \langle \varepsilon \rangle \langle \gamma \rangle \langle \varepsilon \rangle \top \in \mathbb{O}_{rw}$.

The proofs of the following congruence theorems for weak bisimilarity are almost identical to the proofs of the corresponding congruence theorems for delay bisimilarity.

Theorem 6 Let P be a complete standard TSS in weak bisimulation format. Then \leftrightarrow_w is a congruence for P .

Theorem 7 Let P be a complete standard TSS in rooted weak bisimulation format. Then \leftrightarrow_{rw} is a congruence for P .

4.6 Counterexamples

In [9] it was shown that none of the syntactic requirements of the rooted branching bisimulation format in Def. 21 can be omitted, and that the presence of $\aleph \cap \Lambda$ -patience rules is crucial. Here we present a sequence of examples to show that none of the requirements that make up delay resistance is redundant to guarantee that rooted delay bisimilarity is a congruence.

All TSSs in this section are standard, complete, in ready-simulation format and $\aleph \cap \Lambda$ -patient, and their rules are rooted branching bisimulation safe.

Example 4 Let f be a unary function symbol with an \aleph -liquid, Λ -frozen argument, defined by the rule

$$\frac{x \xrightarrow{a}}{f(x) \xrightarrow{b} 0}$$

This rule is positive delay resistant, but fails to be negative delay resistant.

Consider the LTS consisting of the transitions $p_0 \xrightarrow{\tau} p_0$, $p_0 \xrightarrow{a} 0$, $p_1 \xrightarrow{\tau} q$ and $q \xrightarrow{a} 0$. Note that $p_0 \xleftrightarrow{rd} p_1$. However, $f(p_0)$ exhibits no transitions, while $f(p_1) \xrightarrow{b} 0$. So $f(p_0) \not\xleftrightarrow{rd} f(p_1)$. Hence rooted delay bisimilarity is not a congruence.

Example 5 Let f be a unary function symbol with an \aleph -liquid, Λ -frozen argument, defined by the rule

$$\frac{x \xrightarrow{a} y}{f(x) \xrightarrow{b} 0}$$

This TSS is negative delay resistant, but not positive delay resistant.

Consider the LTS from Ex. 4. We have $f(p_0) \xrightarrow{b} 0$, while the process $f(p_1)$ does not exhibit any transitions. So $f(p_0) \not\xleftrightarrow{rd} f(p_1)$. Hence rooted delay bisimilarity is not a congruence.

The following example shows that the requirement that H^d is finite in Def. 17 of positive delay resistance is essential.

Example 6 Let $A = \{a_k \mid k \in \mathbb{Z}_{>0}\} \cup \{b\}$, and let there be binary function symbols f_k for all $k \in \mathbb{Z}_{>0}$, of which both arguments are \aleph -liquid and only the second argument is Λ -liquid. They are defined by the rules

$$\frac{x_1 \xrightarrow{\tau} y}{f_k(x_1, x_2) \xrightarrow{\tau} f_\ell(x_1, y)} \quad (\text{for all } \ell > k) \qquad \frac{x_2 \xrightarrow{\tau} y}{f_k(x_1, x_2) \xrightarrow{\tau} f_k(x_1, y)}$$

$$\frac{\{x_1 \xrightarrow{a_\ell} y_\ell \mid \ell > k\} \cup \{x_2 \xrightarrow{a_k} y_k\}}{f_k(x_1, x_2) \xrightarrow{b} 0}$$

where ω_1 , ω_2 and ω_3 are constants with $\omega_2 \xrightarrow{a_k} 0$ and $\omega_3 \xrightarrow{a_k} 0$ for all $k \geq 1$, and $\omega_1 \xrightarrow{\tau} \omega_3$ and $\omega_2 \xrightarrow{\tau} \omega_3$. Clearly, $\omega_1 \xleftrightarrow{rd} \omega_2$.

With the exception of the rule with infinitely many premises, its rules are delay resistant. That one rule is negative delay resistant, and all its premises are delayable. With regard to Def. 17 it only violates the requirement that H^d needs to be chosen finite. As a result Prop. 2 is violated. Namely, although there are sequences $\omega_1 \xrightarrow{\varepsilon} \xrightarrow{a_\ell}$ for all $\ell \geq 1$, there is no sequence $f_k(\omega_1, \omega_1) \xrightarrow{\varepsilon} \xrightarrow{b}$ for any $k \geq 1$. On the other hand, $f_k(\omega_2, \omega_2) \xrightarrow{b} 0$ for all $k \geq 1$. So rooted delay bisimilarity is not a congruence.

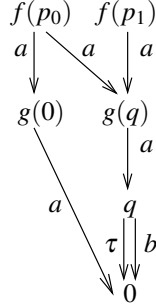
The following example shows that the strengthening of condition 1 to condition 1' in Def. 24 is essential for the rooted weak bisimulation format.

Example 7 Let f be a unary function symbol with an $\aleph \cap \Lambda$ -liquid argument, and g a unary function symbol with an \aleph -frozen, Λ -liquid argument. They are defined by the rules

$$\frac{x \xrightarrow{a} y}{f(x) \xrightarrow{a} g(y)} \qquad \frac{x \xrightarrow{\tau} y}{f(x) \xrightarrow{\tau} f(y)} \qquad \frac{}{g(x) \xrightarrow{a} x}$$

This TSS violates condition 1' of Def. 24 (due to the first rule). On the other hand, it is in rooted delay bisimulation format w.r.t. \aleph and Λ , and thereby delay resistant.

Consider the following LTS, which was already depicted in Sect. 2.1: $p_0 \xrightarrow{a} q$, $p_0 \xrightarrow{a} 0$, $p_1 \xrightarrow{a} q$, $q \xrightarrow{\tau} 0$ and $q \xrightarrow{b} 0$. We have $p_0 \not\leftrightarrow_{rw} p_1$ (but $p_0 \not\leftrightarrow_{rd} p_1$). The LTS rooted in $f(p_0)$ and $f(p_1)$ is as follows.



$f(p_0) \not\leftrightarrow_{rw} f(p_1)$ because the transition $f(p_0) \xrightarrow{a} g(0)$ cannot be mimicked by $f(p_1)$. Hence rooted weak bisimilarity is not a congruence.

5 Checking for delay resistance

On top of the congruence formats for (rooted) delay and weak bisimilarity we have imposed delay resistance, a non-syntactic restriction that is based on concepts from [1, 14]. To show that a TSS P is delay resistant one has to establish a property for each P -ruloid, and a non-trivial TSS has infinitely many of them. This section introduces tools that lighten the burden of checking that a TSS is delay resistant. In particular, syntactic criteria are proposed which imply that a TSS is delay resistant.

5.1 Delay resistance w.r.t. Λ

We introduce the notion “delay resistant w.r.t. Λ ”, in which the conditions of Defs. 16, 17 and 18 need to be checked only for premises containing a variable that occurs only Λ -frozen in the source. We show that delay resistance w.r.t. Λ , together with the $\aleph \cap \Lambda$ -patience rules, conditions 3 and 4 of Def. 21, and one additional condition (see Def. 27) imply delay resistance. So in the context of our congruence formats, when incorporating the additional condition, delay resistance w.r.t. Λ is sufficient to check that a TSS is delay resistant. Moreover, it follows that the delay and weak bisimulation formats do not require delay resistance at all, since with Λ universal there are no Λ -frozen occurrences.

Definition 26 Given a predicate Λ on the arguments of function symbols, a premise $w \xrightarrow{\beta} y$ or $w \xrightarrow{\beta} y$ of a transition rule $\frac{H}{t \xrightarrow{\alpha} u}$ is called Λ -liquid if all variables in w occur Λ -liquid in t . Let H^Λ be the set of Λ -liquid premises in H .

An ntytt rule $\frac{H}{t \xrightarrow{\alpha} u}$ is *positive delay resistant w.r.t. Λ* and a TSS P if there exists a finite set $H^d \subseteq H^+$ of delayable positive premises such that for each set $M \subseteq H^+ \setminus (H^d \cup H^\Lambda)$ there is a rule $r_M = \frac{H_M}{t \xrightarrow{\alpha} u}$, linearly provable from P , where $H_M \subseteq (H \setminus M) \cup M_\tau$.

A rule $\frac{H}{t \xrightarrow{\alpha} u}$ is *negative delay resistant w.r.t. Λ* and P if there is a rule $\frac{H'}{t \xrightarrow{\alpha} u}$, linearly provable from P , with $H' \subseteq H^+ \cup H^{s-} \cup H^\Lambda$.

An ntytt rule $\frac{H}{t \xrightarrow{\alpha} u}$ is *delay resistant w.r.t. Λ and P* if it is positive delay resistant as well as negative delay resistant w.r.t. Λ and P . A standard TSS P in ready simulation format is *delay resistant w.r.t. Λ* if all its linear ruloids with a positive conclusion are delay resistant w.r.t. Λ and \hat{P}^+ .

By taking $\Lambda := \emptyset$ we retrieve the notion of delay resistance from Def. 19.

The syntactic condition in the following definition prevents that a running process x is tested twice.

Definition 27 Given a standard ntytt rule $r = \frac{H}{t \xrightarrow{\alpha} u}$, we define the following syntactic condition:

5. If x has exactly one occurrence in t , which is Λ -liquid, and an \aleph -liquid occurrence in H , then these are the only two occurrences of x in r .

For non-standard rules we take this condition to be vacuously satisfied.

In line with Prop. 3, it can be proved that condition 5 is preserved by the construction of ruloids.

Lemma 6 Let P be a TSS in decent ntyft format, in which each transition rule is rooted branching bisimulation safe and satisfies condition 5 of Def. 27 w.r.t. \aleph and Λ . Then any ntytt rule irredundantly provable from P satisfies condition 5 of Def. 27 w.r.t. \aleph and Λ .

Proof: Let an ntytt rule $\frac{H}{t \xrightarrow{\alpha} u}$ be irredundantly provable from P , by means of a proof π . We prove, using structural induction with respect to π , that this rule satisfies condition 5 of Def. 27 w.r.t. \aleph and Λ .

Induction basis: Suppose π has only one node. Then $\frac{H}{t \xrightarrow{\alpha} u}$ equals $\frac{t \xrightarrow{\alpha} u}{t \xrightarrow{\alpha} u}$ (so t and u are distinct variables). This rule satisfies condition 5 of Def. 27 w.r.t. \aleph and Λ .

Induction step: Let $r \in R$ be the rule and σ the substitution used at the bottom of π . By assumption, r is decent, ntyft, and rooted branching bisimulation safe w.r.t. \aleph and Λ . Let

$$\{v_k \xrightarrow{\beta_k} y_k \mid k \in K\} \cup \{w_\ell \xrightarrow{\gamma_\ell} \mid \ell \in L\}$$

be the set of premises of r , and

$$f(x_1, \dots, x_{ar(f)}) \xrightarrow{\alpha} v$$

the conclusion of r . Then $\sigma(f(x_1, \dots, x_{ar(f)})) = t$ and $\sigma(v) = u$. Moreover, rules $r_k = \frac{H_k}{\sigma(v_k) \xrightarrow{\beta_k} \sigma(y_k)}$ for each $k \in K$ and $r_\ell = \frac{H_\ell}{\sigma(w_\ell) \xrightarrow{\gamma_\ell}}$ for each $\ell \in L$ are irredundantly provable from P by means of strict subproofs of π , where $H = \bigcup_{k \in K} H_k \cup \bigcup_{\ell \in L} H_\ell$.

As r is decent, $\text{var}(v_k) \subseteq \{x_1, \dots, x_{ar(f)}\}$, so $\text{var}(\sigma(v_k)) \subseteq \text{var}(t)$ for each $k \in K$. Likewise, $\text{var}(\sigma(w_\ell)) \subseteq \text{var}(t)$ for each $\ell \in L$. From $\text{rhs}(H) \cap \text{var}(t) = \emptyset$ it follows that $\text{rhs}(H_k) \cap \text{var}(\sigma(v_k)) = \emptyset$ for each $k \in K$, and $\text{rhs}(H_\ell) \cap \text{var}(\sigma(w_\ell)) = \emptyset$ for each $\ell \in L$. So for each $k \in K$ and $\ell \in L$, the rules r_k and r_ℓ are ntytt rules. By Lem. 1, they are decent. By Lem. 4 they are rooted branching bisimulation safe w.r.t. \aleph and Λ . And by induction, they satisfy condition 5 of Def. 27 w.r.t. \aleph and Λ .

Suppose that x has exactly one occurrence in t , which is Λ -liquid, and an \aleph -liquid occurrence in H . Then there is an $i_0 \in \{1, \dots, ar(f)\}$ with $\Lambda(f, i_0)$ such that x has exactly one occurrence in $\sigma(x_{i_0})$, which is Λ -liquid; moreover, $x \notin \text{var}(\sigma(x_i))$ for each $i \neq i_0$. And x occurs \aleph -liquid in the left-hand side of a premise in H_m for some $m \in K \cup L$. Since r_m is rooted branching bisimulation safe w.r.t. \aleph and Λ , by condition 3 of Def. 21 or Def. 23, x must occur \aleph -liquid in $\sigma(v_m)$ or $\sigma(w_m)$. By the decency of r , this implies that x_{i_0} occurs \aleph -liquid in v_m or w_m . Since r is rooted branching bisimulation safe w.r.t. \aleph and Λ , and $\Lambda(f, i_0)$, by condition 5 of Def. 27, this is the only occurrence of x_{i_0} in the v_k or w_ℓ for each $k \in K$ and

$\ell \in L$, and $x_{i_0} \notin \text{var}(v)$. Moreover, either by condition 3 (if $\neg \aleph(f, i_0)$) or condition 4 (if $\aleph(f, i_0)$), using that $\Lambda(f, i_0)$ of Def. 21, $m \in K$. And by condition 2 of Def. 21, the occurrence of x_{i_0} in v_m is Λ -liquid. It follows that x has exactly one occurrence in $\sigma(v_m)$, which is Λ -liquid, and that x does not occur in $\sigma(v_k)$ for each $k \in K \setminus \{m\}$ and $\sigma(w_\ell)$ for each $\ell \in L$. So x does not occur in r_k for each $k \in K \setminus \{m\}$ and r_ℓ for each $\ell \in L$, in view of the decency of these rules, and the fact that x cannot occur in $\text{rhs}(H)$. That is, x does not occur in H_k and $\sigma(y_k)$ for each $k \in K \setminus \{m\}$, and H_ℓ for each $\ell \in L$. And since r_m satisfies condition 5 of Def. 27, while x occurs \aleph -liquid in H_m and has exactly one occurrence in $\sigma(v_m)$, which is Λ -liquid, x has only one occurrence in H_m , and $x \notin \text{var}(\sigma(y_m))$. Concluding, x has only one occurrence in H ; and since $x_{i_0} \notin \text{var}(v)$, $x \notin \text{var}(\sigma(x_i))$ for each $i \neq i_0$, and $x \notin \text{var}(\sigma(y_k))$ for each $k \in K$, it follows that $x \notin \text{var}(\sigma(v))$. \square

Corollary 1 Let P be a standard TSS in ready simulation format, in which each rule is rooted branching bisimulation safe and satisfies condition 5 of Def. 27 w.r.t. \aleph and Λ . Then any P -ruloid satisfies condition 5 of Def. 27 w.r.t. \aleph and Λ .

Proof: We recall from Sect. 2.6, that the standard TSS P can be transformed into a TSS P^+ in decent ntyft format; the P -ruloids are those decent nxytt rules that are irredundantly provable from P^+ .

As the rules of P are rooted branching bisimulation safe and satisfying condition 5 of Def. 27 w.r.t. \aleph and Λ , then so are the rules in P^+ . Namely, as described in Sect. 2.6, the rules in P^+ are constructed in three steps. The first step (the conversion of P to decent ntyft format) clearly preserves the rooted branching bisimulation format, as well as condition 5. The second step (the construction to reduce left-hand sides of positive premises to variables) yields an intermediate TSS, all of whose rules are irredundantly provable from P , and thus is covered by Lem. 4 for rooted branching bisimulation safety, and by Lem. 6 for condition 5. The the final step adds rules with negative conclusions to the TSS; as pointed out in the proof of Prop. 3 (in [9]), these added rules are also rooted branching bisimulation safe. Trivially, they satisfy condition 5.

Since the rules in P^+ are rooted branching bisimulation safe and satisfy condition 5 w.r.t. \aleph and Λ , by Lem. 6, each P -ruloid satisfies condition 5 w.r.t. \aleph and Λ . \square

The next lemma is the crucial step in showing that delay resistance w.r.t. Λ , together with the $\aleph \cap \Lambda$ -patience rules, conditions 3 and 4 of Def. 21, and condition 5 of Def. 27, implies delay resistance.

Lemma 7 Let P be an $\aleph \cap \Lambda$ -patient TSS, and let $r = \frac{H}{t \xrightarrow{\alpha} u}$ be an xyntt rule with t univariate, linearly provable from P , that is rooted branching bisimulation safe and satisfies condition 5 of Def. 27 w.r.t. \aleph and Λ . If r is delay resistant w.r.t. Λ and P , then it is delay resistant w.r.t. P .

Proof: To show that r is positive delay resistant w.r.t. P it suffices to show that $H^\Lambda \cap H^+$ is finite, and that all premises in $H^\Lambda \cap H^+$ are delayable. To show that r is negative delay resistant w.r.t. P it suffices to show that H^Λ does not contain negative premises at all.

By definition, for each premise $x \xrightarrow{\beta} y$ or $x \xrightarrow{\beta} \cdot$ in H^Λ , x occurs Λ -liquid in t . By condition 3 of Def. 21, the (unique) occurrence of x in t is also \aleph -liquid. By condition 4 of Def. 21, H^Λ contains only one premise with left-hand side x , which must be positive. So H^Λ contains no negative premises. And since $\text{var}(t)$ is finite, it follows that $H^\Lambda \cap H^+$ is finite too.

Let $H = H_0 \uplus \{x \xrightarrow{\beta} y\}$ where x occurs $\aleph \cap \Lambda$ -liquid in t . We need to show that there exist xyntt rules $\frac{H_1}{t \xrightarrow{\tau} v}$ and $\frac{H_2}{v \xrightarrow{\alpha} u}$, linearly provable from P , with $H_1 \subseteq H_0 \cup \{x \xrightarrow{\tau} z\}$ and $H_2 \subseteq H_0 \cup \{z \xrightarrow{\beta} y\}$ for some term v and fresh variable z . By condition 5 of Def. 27, x does not occur in u or H_0 . Let v be obtained

from t by substituting a fresh variable z for x . Then $\frac{H_2}{v \xrightarrow{\alpha} u}$ is a substitution instance of $\frac{H}{t \xrightarrow{\alpha} u}$, and hence linearly provable from P , using Lem. 2. As $\frac{x \xrightarrow{\tau} z}{t \xrightarrow{\tau} v}$ is $\aleph \cap \Lambda$ -patient, it is also linearly provable from P . \square

The following lemmas ensure that to verify delay resistance of a TSS P , it suffices to check delay resistance w.r.t. P for linear P -ruloids with a univariate source.

Lemma 8 Let P be a TSS in ntyft format. Any ntytt rule linearly provable from P is the instance under a substitution $\sigma : V \rightarrow V$ of a rule, linearly provable from P , with a univariate source t . Moreover, $\text{dom}(\sigma) = \text{var}(t)$.

Proof: Straightforward by induction on the linear proof of the rule. \square

Lemma 9 Let $r = \frac{H}{t \xrightarrow{\alpha} u}$ be an ntytt rule, and $\sigma : \text{var}(t) \rightarrow V$.

If r is delay resistant w.r.t. a TSS P , then so is the rule $\sigma(r) = \frac{\sigma(H)}{\sigma(t) \xrightarrow{\alpha} \sigma(u)}$.

Proof: Let r be delay resistant w.r.t. P . Since r is negative delay resistant w.r.t. P , there exists a rule $\frac{H'}{t \xrightarrow{\alpha} u}$, linearly provable from P , with $H' \subseteq H^+ \cup H^{s-}$. By Lem. 2, the rule $\frac{\sigma(H')}{\sigma(t) \xrightarrow{\alpha} \sigma(u)}$ is also linearly provable from P , and $\sigma(H') \subseteq \sigma(H)^+ \cup \sigma(H)^{s-}$. It follows that $\sigma(r)$ is negative delay resistant w.r.t. P .

As σ does not affect $\text{rhs}(H)$, σ is injective on H^+ . Let $H^d \subseteq H^+$ be the finite set of delayable premises that exists by Def. 17. Take $\sigma(H)^d := \sigma(H^d)$. Then any subset of $\sigma(H)^+ \setminus \sigma(H)^d$ can be written as $\sigma(M)$ with $M \subseteq H \setminus H^d$. By Def. 17 there exists a rule $r_M = \frac{H_M}{t \xrightarrow{\alpha} u}$, linearly provable from P , where $H_M \subseteq (H \setminus M) \cup M_\tau$. Hence the rule $\sigma(r_M) = \frac{\sigma(H_M)}{\sigma(t) \xrightarrow{\alpha} \sigma(u)}$, where $\sigma(H_M) \subseteq (\sigma(H) \setminus \sigma(M)) \cup \sigma(M)_\tau$, is linearly provable from P by Lem. 2.

It remains to show that the premises in $\sigma(H^d)$ are delayable. This follows immediately from the delayability of the premises in H^d , by applying σ . \square

The next proposition states that delay resistance w.r.t. Λ , together with the presence of the $\aleph \cap \Lambda$ -patience rules, rooted branching bisimulation safeness and condition 5 of Def. 27, is sufficient to guarantee delay resistance.

Proposition 9 Let P be a standard TSS in ready simulation format, in which each transition rule is rooted branching bisimulation safe and satisfies condition 5 of Def. 27 w.r.t. \aleph and Λ . Let moreover P be $\aleph \cap \Lambda$ -patient and delay resistant w.r.t. Λ . Then P is delay resistant.

Proof: Let $r = \frac{H}{t \xrightarrow{\alpha} u}$ be a linear P -ruloid, i.e. an nxytt rule, linearly provable from the TSS \hat{P}^+ constructed in Sec. 2.6. We need to show that r is delay resistant. Since P is $\aleph \cap \Lambda$ -patient, so is \hat{P}^+ . Using Lemmas 8 and 9 (and that \hat{P}^+ is in ntyft format) we may restrict attention to the case that t is univariate. By Prop. 3, r is rooted branching bisimulation safe w.r.t. \aleph and Λ , and by Cor. 1 it moreover satisfies condition 5 of Def. 27 w.r.t. \aleph and Λ . By assumption, r is delay resistant w.r.t. Λ and P . The result now follows from Lem. 7. \square

If Λ is universal, all premises are Λ -liquid. Hence in the presence of condition 5 of Def. 27, the requirement of delay resistance can be dropped from the delay and weak bisimulation formats.

Definition 28 A standard TSS P is in *syntactic delay bisimulation format* if it is in ready simulation format, and, for some \aleph and the universal predicate Λ , it is $\aleph \cap \Lambda$ -patient and all its transition rules are rooted branching bisimulation safe and satisfy condition 5 of Def. 27 w.r.t. \aleph and Λ .

The *syntactic weak bisimulation format* is defined likewise, but using condition 1' of Def. 24 instead of condition 1 of Def. 21.

Corollary 2 Let P be a complete standard TSS in syntactic delay bisimulation format. Then \leftrightarrow_d (as well as \leftrightarrow_{rd}) is a congruence for P .

Let P be a complete standard TSS in syntactic weak bisimulation format. Then \leftrightarrow_w (as well as \leftrightarrow_{rw}) is a congruence for P .

5.2 Semi-syntactic criteria for delay resistance

We now introduce requirements on the rules of a TSS P that imply delay resistance of P . This yields what could be called *semi-syntactic* congruence formats. They are not purely syntactic, because one of the conditions (in Def. 29) requires the existence of certain linearly provable rules; however, all conditions need to be checked for rules in P only (rather than for P -ruloids).

Definition 29 A premise $w \xrightarrow{\beta} y$ of an ntytt rule $r = \frac{H}{t \xrightarrow{\alpha} u}$ is *manifestly delayable* in a TSS $P = (\Sigma, R)$ if, for some term v and fresh variable z , there is a transition rule $\frac{H_1}{t \xrightarrow{\tau} v}$ in \bar{R} , as well as an ntytt rule $\frac{H_2}{v \xrightarrow{\alpha} u}$ linearly provable from P , with $H_1 \subseteq (H \setminus \{w \xrightarrow{\beta} y\}) \cup \{w \xrightarrow{\tau} z\}$ and $H_2 \subseteq (H \setminus \{w \xrightarrow{\beta} y\}) \cup \{z \xrightarrow{\beta} y\}$.

Here \bar{R} denotes the set R of transition rules up to a bijective renaming of variables. The difference with Def. 16 is that here the rule $\frac{H_1}{t \xrightarrow{\tau} v}$ needs to be in \bar{R} , rather than merely being linearly provable from P . So clearly each manifestly delayable premise is delayable.

Definition 30 A transition rule $\frac{H}{t \xrightarrow{\alpha} u}$ is *manifestly negative delay resistant*, or more briefly *negative-stable*, if for every premise $w \xrightarrow{\gamma} _$ in H , also $w \xrightarrow{\gamma} _$ is in H . A TSS is *negative-stable* if all its rules are.

The difference with Def. 18 is that here the requirement also applies to redundant premises $w \xrightarrow{\alpha} _$. Clearly each negative-stable rule is negative delay resistant w.r.t. any TSS.

Definition 31 An ntytt rule $\frac{H}{t \xrightarrow{\alpha} u}$ is *manifestly delay resistant* w.r.t. [a predicate Λ and] a TSS $P = (\Sigma, R)$ if it is negative-stable and there exists a finite set $H^d \subseteq H^+$ of manifestly delayable positive premises, such that for each set $M \subseteq H^+ \setminus (H^d \cup H^\Lambda)$ there is a rule $r_M = \frac{H_M}{t \xrightarrow{\alpha} u}$ in \bar{R} , where $H_M \subseteq (H \setminus M) \cup M_\tau$.

Again, the rule r_M needs to be in \bar{R} , rather than merely being linearly provable from P . The material in this section and in the appendix comes in two flavours: incorporating a predicate Λ on arguments of function symbols, or omitting it. The latter is equivalent to taking $\Lambda = \emptyset$. Notationally, we will capture both by putting the optional material, pertaining to Λ , between square brackets.

Clearly, a manifestly delay resistant rule w.r.t. [Λ and] P is delay resistant w.r.t. [Λ and] P (cf. Def. 19 [or Def. 26]).

Definition 32 A standard TSS P in decent ntyft format is *manifestly delay resistant* [w.r.t. Λ] if all its transition rules are manifestly delay resistant w.r.t. [Λ and] P . A standard TSS P in ready simulation format is *manifestly delay resistant* [w.r.t. Λ] if its conversion P^\dagger to decent ntyft format (see Sec. 2.6) is manifestly delay resistant [w.r.t. Λ].

Note that in contrast to the notion of a delay resistant TSS from Def. 19, here the property is only required for the rules of P , instead of all linear P -ruloids. The following theorem, whose proof is presented in Appendix B, provides a semi-syntactic version of all our congruence formats.

Theorem 8 Any manifestly delay resistant standard TSS in ready simulation format is delay resistant.

The following variant of this theorem, whose proof is also presented in Appendix B, mixes in the insights of Sec. 5.1, and provides semi-syntactic versions of our congruence formats that are normally easier to apply.

Theorem 9 Let P be a standard TSS in ready simulation format, in which each transition rule is rooted branching bisimulation safe and satisfies condition 5 of Def. 27 w.r.t. \mathfrak{K} and Λ . Let moreover P be $\mathfrak{K} \cap \Lambda$ -patient and manifestly delay resistant w.r.t. Λ . Then P is delay resistant.

Definition 33 A standard TSS P is in *manifest rooted delay bisimulation format* if it is in ready simulation format, and, for some \mathfrak{K} and Λ , it is $\mathfrak{K} \cap \Lambda$ -patient and manifestly delay resistant w.r.t. Λ , and it only contains transition rules that are rooted branching bisimulation safe and satisfy condition 5 of Def. 27 w.r.t. \mathfrak{K} and Λ .

The *manifest rooted weak bisimulation format* is defined likewise, but using condition 1' of Def. 24 instead of condition 1 of Def. 21.

Corollary 3 Let P be a complete standard TSS in manifest rooted delay bisimulation format. Then \leftrightarrow_{rd} is a congruence for P .

Let P be a complete standard TSS in manifest rooted weak bisimulation format. Then \leftrightarrow_{rw} is a congruence for P .

For most applications, to check delay resistance it suffices to check the following, simpler property.

Definition 34 A standard ntytt rule is *simply Λ -delay resistant* in a TSS P if it is negative-stable and has finitely many positive premises, all of which are either manifestly delayable in P or Λ -liquid. A standard TSS in decent ntytt format is *simply Λ -delay resistant* if all its transition rules are.

Clearly, a simply Λ -delay resistant transition rule $r = \frac{H}{t \xrightarrow{\alpha} u}$ is manifestly delay resistant w.r.t. Λ , by taking $H^d := H^+ \setminus H^\Lambda$ and $r_\emptyset := r$.

5.3 Syntactic criteria for delay resistance

We show how delay resistance can be replaced by additional syntactic requirements. Def. 22 is adapted as follows. On the one hand the requirement that the TSS is delay resistant is dropped. On the other hand, rules must satisfy condition 5 of Def. 27, and the TSS must be in nxytt format and negative-stable. And there are additional syntactic restrictions if a Λ -frozen argument of the source is tested in the premises (condition 3). Furthermore, there is a syntactic requirement with regard to predicates $\Delta_\alpha \subseteq \mathfrak{K} \cap \Lambda$ (condition 4).

Definition 35 A standard TSS $P = (\Sigma, R)$ is in *syntactic rooted delay bisimulation format* if, for some \mathfrak{K} and Λ and predicates $\Delta_\alpha \subseteq \mathfrak{K} \cap \Lambda$ where α ranges over $A \cup \{\tau\}$:

1. P is in decent nxytt format and $\mathfrak{K} \cap \Lambda$ -patient.

2. Each rule in R is rooted branching bisimulation safe and negative-stable, satisfies condition 5 of Def. 27 w.r.t. \aleph and Λ , and has finitely many positive premises.
3. If R contains a rule $\frac{H \uplus \{x_i \xrightarrow{\beta} y\}}{f(x_1, \dots, x_{ar(f)}) \xrightarrow{\alpha} u}$ where $\neg \Lambda(f, i)$, then:
 - (a) $\beta = \alpha$;
 - (b) R contains a rule $\frac{H' \uplus \{x_i \xrightarrow{\tau} y\}}{f(x_1, \dots, x_{ar(f)}) \xrightarrow{\tau} u}$ with $H' \subseteq H$; and
 - (c) y has exactly one, Δ_α -liquid occurrence in u .
4. If $\Delta_\alpha(f, i)$, then R contains $\frac{x_i \xrightarrow{\alpha} y}{f(x_1, \dots, x_i, \dots, x_{ar(f)}) \xrightarrow{\alpha} f(x_1, \dots, y, \dots, x_{ar(f)})}$.

P is in *syntactic rooted weak bisimulation format* if its rules moreover satisfy condition 1' of Def. 27.

The introduction of predicates $\Delta_\alpha \subseteq \aleph \cap \Lambda$ is of practical importance. If in Def. 35 one would replace the occurrences of Δ_α by $\aleph \cap \Lambda$, then for instance the *encapsulation* operator ∂_H , which blocks all actions in the set H , would violate condition 4 of Def. 35. Namely, the argument of ∂_H is $\aleph \cap \Lambda$ -liquid, but there is no rule $\frac{x \xrightarrow{\alpha} y}{\partial_H(x) \xrightarrow{\alpha} \partial_H(y)}$ if $a \in H$.

Actually, in many applications Δ_α can be empty, as a rule that tests an \aleph -liquid, Λ -frozen argument of the source in practice tends to have a single y as right-hand side of the conclusion, so that condition 3c of Def. 35 is trivially satisfied; a notable example is the rule $\frac{x_1 \xrightarrow{\alpha} y}{x_1 + x_2 \xrightarrow{\alpha} y}$ for alternative composition.

Proposition 10 Let P be a TSS in syntactic rooted delay bisimulation format. Then it is in manifest rooted delay bisimulation format.

Proof: Let \aleph , Λ and Δ_γ for all $\gamma \in A \cup \{\tau\}$ be such that P satisfies the restrictions in Def. 35. It suffices to show that P is simply Λ -delay resistant. Consider a rule $\frac{H \uplus \{x_i \xrightarrow{\beta} y\}}{f(x_1, \dots, x_n) \xrightarrow{\alpha} u}$ of P with $\neg \Lambda(f, i)$. By condition 3a of Def. 35, $\beta = \alpha$. And by condition 3c of Def. 35, y has exactly one, Δ_α -liquid occurrence in u . It suffices to show that $x_i \xrightarrow{\beta} y$ is manifestly delayable in P . So, for some term v and fresh variable z , there must be a rule $\frac{H_1}{f(x_1, \dots, x_n) \xrightarrow{\tau} v}$ in \bar{R} , as well as a rule $\frac{H_2}{v \xrightarrow{\alpha} u}$ linearly provable from P , with $H_1 \subseteq H \cup \{x_i \xrightarrow{\tau} z\}$ and $H_2 \subseteq H \cup \{z \xrightarrow{\beta} y\}$. Let v be obtained by substituting z for y in u . The first of these rules exists by condition 3b of Def. 35, substituting z for y . The second is the rule $\frac{z \xrightarrow{\beta} y}{v \xrightarrow{\alpha} u}$, which can be derived by condition 4 of Def. 35. Here we use that $\beta = \alpha$ and y has exactly one, Δ_α -liquid occurrence in u . \square

Prop. 10, together with Cor. 3, gives rise to the following corollary.

Corollary 4 Let the complete standard TSS P be in syntactic rooted delay bisimulation format. Then \leftrightarrow_{rd} is a congruence for P .

Let the complete standard TSS P be in syntactic rooted weak bisimulation format. Then \leftrightarrow_{rw} is a congruence for P .

6 Applications

In this section we revisit some applications of our congruence formats that were already considered in [9]: the basic process algebra $\text{BPA}_{\varepsilon\delta\tau}$, extended with binary Kleene star as an example where the predicates Δ_α from Def. 35 are non-empty, and initial priority because it includes negative premises. We also consider a deadlock test that is outside the syntactic rooted delay bisimulation format. In all these cases our formats provide congruence results for rooted delay and weak bisimilarity, while they are outside the congruence formats for rooted delay and weak bisimilarity from [1, 14].

The TSSs in this section are all $\mathfrak{X} \cap \Lambda$ -patient and in decent xynft format.

6.1 Basic process algebra

Consider the basic process algebra $\text{BPA}_{\varepsilon\delta\tau}$, consisting of: constants from an alphabet $\text{Act} \cup \{\tau\}$; the empty process ε ; the deadlock δ ; alternative composition $t_1 + t_2$; and sequential composition $t_1 \cdot t_2$. Let ℓ range over $\text{Act} \cup \{\tau\}$ and α over $\text{Act} \cup \{\tau, \surd\}$. The transition rules are:

$$\begin{array}{c} \frac{}{\ell \xrightarrow{\ell} \varepsilon} \quad \frac{}{\varepsilon \xrightarrow{\surd} \delta} \quad \frac{x_1 \xrightarrow{\alpha} y}{x_1 + x_2 \xrightarrow{\alpha} y} \quad \frac{x_2 \xrightarrow{\alpha} y}{x_1 + x_2 \xrightarrow{\alpha} y} \\ \\ \frac{x_1 \xrightarrow{\ell} y}{x_1 \cdot x_2 \xrightarrow{\ell} y \cdot x_2} \quad \frac{x_1 \xrightarrow{\surd} y_1 \quad x_2 \xrightarrow{\alpha} y_2}{x_1 \cdot x_2 \xrightarrow{\alpha} y_2} \end{array}$$

To show that \leftrightarrow_{rd} and \leftrightarrow_{rw} are congruences, we argue that this TSS satisfies the conditions of Def. 35. In [9] it was shown that it is in rooted η -bisimulation format, with \mathfrak{X} and Λ defined as follows. Since the arguments of alternative and sequential composition can all execute immediately, \mathfrak{X} holds for all these arguments. Since only the first argument of sequential composition can contain running processes, it is the only argument for which Λ holds. Since the TSS is positive, it surely is negative-stable. With regard to condition 2 of Def. 35, we still need to check that the rules satisfy condition 5 of Def. 27: only the two rules for sequential composition contain a Λ -liquid occurrence of a variable, x_1 , in their source; and in both cases x_1 has only one other occurrence in the rule, in the left-hand side of a premise. Condition 3 of Def. 35 needs to be verified with regard to the two rules for alternative composition and the second rule for sequential composition, since in these rules a Λ -frozen argument of the source is tested in a premise. It is not hard to see that condition 3 is satisfied for these rules, where we can take $\Delta_\gamma = \emptyset$ for all γ . Hence condition 4 of Def. 35 is trivially satisfied.

Concluding, by Cor. 4 rooted delay and weak bisimilarity are congruences for $\text{BPA}_{\varepsilon\delta\tau}$.

6.2 Binary Kleene star

The *binary Kleene star* $t_1^*t_2$ [18] repeatedly executes t_1 until it executes t_2 . This operational behaviour is captured by the following rules, which are added to the rules for $\text{BPA}_{\varepsilon\delta\tau}$.

$$\frac{x_1 \xrightarrow{\ell} y}{x_1^*x_2 \xrightarrow{\ell} y \cdot (x_1^*x_2)} \quad \frac{x_2 \xrightarrow{\alpha} y}{x_1^*x_2 \xrightarrow{\alpha} y}$$

Again, to show that \leftrightarrow_{rd} and \leftrightarrow_{rw} are congruences, we argue that the resulting TSS satisfies the conditions of Def. 35. In [9] it was shown that it is in rooted η -bisimulation format, if we take the arguments of the binary Kleene star to be Λ -frozen (they do not contain running processes) and \mathfrak{X} -liquid (they can start executing immediately). Since the arguments of the binary Kleene star are Λ -frozen, condition con-

dition 5 of Def. 27 is trivially satisfied. Condition 3 of Def. 35 needs to be verified for the two rules for binary Kleene star. It is easy to see that conditions 3(a,b) are satisfied by both rules, and that the second rule for binary Kleene star trivially satisfies condition 3(c). In view of the latter condition with regard to the first rule for binary Kleene star, we mark the first argument of sequential composition by Δ_ℓ for all $\ell \in Act \cup \{\tau\}$. No other arguments are marked by the Δ_γ . It is easy to see that condition 4 of Def. 35 is satisfied with respect to the Δ_γ . (Note that for this last condition it is essential that the first argument of sequential composition is not marked by Δ_γ .)

Concluding, by Cor. 4 rooted delay and weak bisimilarity are congruences for $BPA_{\varepsilon\delta\tau}$ with the binary Kleene star.

6.3 Initial priority

Initial priority is a unary function that assumes an ordering on atomic actions. The term $\theta(t)$ executes the transitions of t , with the restriction that an initial transition $t \xrightarrow{\ell} t_1$ only gives rise to an initial transition $\theta(t) \xrightarrow{\ell} t_1$ if there does not exist an initial transition $t \xrightarrow{\ell'} t_2$ with $\ell < \ell'$. This intuition is captured by the first rule for the initial priority operator below, which is added to the rules for $BPA_{\varepsilon\delta\tau}$.

$$\frac{x \xrightarrow{\ell} y \quad x \not\xrightarrow{\ell'} \text{ for all } \ell' > \ell}{\theta(x) \xrightarrow{\ell} y} \qquad \frac{x \xrightarrow{\checkmark} y}{\theta(x) \xrightarrow{\checkmark} y}$$

We take the argument of initial priority to be Λ -frozen (it does not contain running processes) and \aleph -liquid (it can start executing immediately). In [9] it was observed that the resulting TSS is in rooted η -bisimulation format, irrespective of the ordering on atomic actions.

If we take τ to be greater than all atomic actions in Act , then both rules are negative-stable, because instances of the first rule for initial priority with a premise $x \xrightarrow{a} y$ for some $a \in Act$ are guaranteed to also contain the premise $x \xrightarrow{\tau} y$. In fact it is sufficient to require $\forall \ell : (\exists \ell' : \ell' > \ell) \Rightarrow \tau > \ell$.

To show that \xrightarrow{rd} and \xrightarrow{rw} are congruences, we argue that the TSS satisfies the conditions of Def. 35. Condition 5 of Def. 27 is trivially satisfied by the rules for initial priority, because its argument is Λ -frozen. Condition 3 of Def. 35 needs to be verified with regard to the two rules for initial priority, since in these rules the Λ -frozen argument of the source is tested in a premise. It is not hard to see that condition 3 is satisfied for these rules, where we can take $\Delta_\gamma = \emptyset$ for all γ . In particular, condition 3(b) is satisfied by the first rule for initial priority, because this rule with $\ell = \tau$ contains no negative premises. Since the Δ_γ are empty, condition 4 of Def. 35 is trivially satisfied.

Concluding, if τ is greater than all atomic actions in Act , rooted delay and weak bisimilarity are congruences for $BPA_{\varepsilon\delta\tau}$ with initial priority.

We note that if τ is smaller than some atomic action a in Act , then rooted delay and weak bisimilarity are not congruences for $BPA_{\varepsilon\delta\tau}$ with initial priority. For example, consider $\tau \cdot a$ and $(\tau \cdot a) + a$. These process terms are rooted delay bisimilar. However, the transition $\theta(\tau \cdot a) \xrightarrow{\tau} a$ cannot be mimicked by $\theta((\tau \cdot a) + a)$, as the latter term can only perform an a -transition to ε . So these terms are not rooted weakly bisimilar.

6.4 Deadlock testing

Finally we give an example that is outside the format from Def. 35, but that is covered by the more general format induced by Thm. 8. Let $yes, no \in Act$. The unary operator f , defined by the following two rules, tests whether its argument is a deadlock.

$$\frac{x \xrightarrow{\alpha} y}{f(x) \xrightarrow{no} \delta} \qquad \frac{x \not\xrightarrow{\alpha} y \text{ for all } \alpha}{f(x) \xrightarrow{yes} \delta}$$

The argument of f is Λ -frozen and \aleph -liquid. Clearly the first rule violates condition 3 of Def. 35.

The TSS is complete, in rooted η -bisimulation format and manifestly delay resistant. In particular, for both rules, $H^d := \emptyset$, while r_\emptyset is the rule itself. Furthermore, for the first rule, $r_{\{x \xrightarrow{\alpha} y\}}$ is $\frac{x \xrightarrow{\tau} y}{f(x) \xrightarrow{no} \delta}$. So according to Thm. 8 the TSS is delay resistant. Hence, by Thms. 5 and 7, rooted delay and weak bisimilarity are congruences for $\text{BPA}_{\varepsilon\delta\tau}$ with deadlock testing.

7 Conclusions

We have extended the method from [9] for modal decomposition and the derivation of congruence formats so that it applies to delay and weak bisimilarity. This research line gives a deeper insight into the link between modal logic and structural operational semantics, and provides a framework for the derivation of congruence formats for the spectrum of weak semantics from [12].

Admittedly, the whole story is quite technical and intricate. Partly this is because we build on a rich body of earlier work in the realm of structural operational semantics: the notions of well-supported proofs and complete TSSs from [13] (or actually [11] in logic programming); the ntyft format from [4, 15]; the transformation to ruloids, which for the main part goes back to [6]; and the work on modal decomposition and congruence formats from [2] and [9].

In spite of these technicalities, we have arrived at a relatively simple framework for the derivation of congruence formats for weak semantics. Namely, for this one only needs to: (1) provide a modal characterisation of the weak semantics under consideration; (2) study the class of modal formulas that result from decomposing this modal characterisation, and formulate syntactic restrictions on TSSs to bring this class of modal formulas within the original modal characterisation; and (3) check that these syntactic restrictions are preserved under the transformation to ruloids. As shown in Sect. 4.5, steps (2) and (3) are very similar in structure for delay and weak bisimilarity. And as said, the end results are congruence formats that are more general and at the same time more elegant than existing congruence formats for these semantics in the literature.

The door is now open to derive congruence formats for a wide range of weak semantics. However, further work is needed to cover the entire spectrum in [12]. Specifically, it would be interesting to extend the current framework to divergence-sensitive semantics. For future research, it would also be interesting to see whether the bridge between modal logic and congruence formats could be employed in the realm of logics and semantics for e.g. probabilities and security. As a first step in this direction, in [10] the decomposition method for Hennessy-Milner logic was lifted to probabilistic systems.

References

- [1] B. BLOOM (1995): *Structural operational semantics for weak bisimulations*. *Theoretical Computer Science* 146(1/2), pp. 25–68.
- [2] B. BLOOM, W.J. FOKKINK & R.J. VAN GLABBEEK (2004): *Precongruence formats for decorated trace semantics*. *ACM Transactions on Computational Logic* 5(1), pp. 26–78.
- [3] B. BLOOM, S. ISTRAIL & A.R. MEYER (1995): *Bisimulation can't be traced*. *Journal of the ACM* 42(1), pp. 232–268.

- [4] R.N. BOL & J.F. GROOTE (1996): *The meaning of negative premises in transition system specifications*. *Journal of the ACM* 43(5), pp. 863–914.
- [5] W.J. FOKKINK (2000): *Rooted branching bisimulation as a congruence*. *Journal of Computer and System Sciences* 60(1), pp. 13–37.
- [6] W.J. FOKKINK & R.J. VAN GLABBEK (1996): *Ntyft/ntyxt rules reduce to ntree rules*. *Information and Computation* 126(1), pp. 1–10.
- [7] W.J. FOKKINK & R.J. VAN GLABBEK (2016): *Divide and Congruence II: Delay and Weak Bisimilarity*. To appear in *Proc. LICS'16*. ACM/IEEE.
- [8] W.J. FOKKINK, R.J. VAN GLABBEK & P. DE WIND (2006): *Compositionality of Hennessy-Milner logic by structural operational semantics*. *Theoretical Computer Science* 354(3), pp. 421–440.
- [9] W.J. FOKKINK, R.J. VAN GLABBEK & P. DE WIND (2012): *Divide and congruence: From decomposition of modal formulas to preservation of branching and η -bisimilarity*. *Information and Computation* 214, pp. 59–85.
- [10] D. GEBLER & W.J. FOKKINK (2012): *Compositionality of probabilistic Hennessy-Milner logic through structural operational semantics*. In *Proc. CONCUR'12*, LNCS 7454, pp. 395–409. Springer.
- [11] A. VAN GELDER, K. ROSS & J.S. SCHLIPF (1991): *The well-founded semantics for general logic programs*. *Journal of the ACM* 38(3), pp. 620–650.
- [12] R.J. VAN GLABBEK (1993): *The linear time-branching time spectrum II: The semantics of sequential systems with silent moves*. In *Proc. CONCUR'93*, LNCS 715, pp. 66–81. Springer.
- [13] R.J. VAN GLABBEK (2004): *The meaning of negative premises in transition system specifications II*. *Journal of Logic and Algebraic Programming* 60/61, pp. 229–258.
- [14] R.J. VAN GLABBEK (2011): *On cool congruence formats for weak bisimulations*. *Theoretical Computer Science*, 412(28):3283–3302.
- [15] J.F. GROOTE (1993): *Transition system specifications with negative premises*. *Theoretical Computer Science* 118(2), pp. 263–299.
- [16] J.F. GROOTE & F.W. VAANDRAGER (1992): *Structured operational semantics and bisimulation as a congruence*. *Information and Computation* 100(2), pp. 202–260.
- [17] M. HENNESSY & R. MILNER (1985): *Algebraic laws for non-determinism and concurrency*. *Journal of the ACM* 32(1), pp. 137–161.
- [18] S.C. KLEENE (1956): *Representation of events in nerve nets and finite automata*. In (C. Shannon and J. McCarthy, eds.) *Automata Studies*, pp. 3–41. Princeton University Press.
- [19] K.G. LARSEN & X. LIU (1991): *Compositionality through an operational semantics of contexts*. *Journal of Logic and Computation* 1(6), pp. 761–795.
- [20] R. MILNER (1981): *A modal characterisation of observable machine-behaviour*. In *Proc. CAAP'81*, LNCS 112, pp. 25–34. Springer.
- [21] R. MILNER (1989): *Communication and Concurrency*. Prentice Hall.
- [22] G.D. PLOTKIN (2004): *A structural approach to operational semantics*. *Journal of Logic and Algebraic Programming* 60/61, pp. 17–139. Originally appeared in 1981.
- [23] R. DE SIMONE (1985): *Higher-level synchronising devices in MEIJE-SCCS*. *Theoretical Computer Science* 37(3), pp. 245–267.
- [24] I. ULIDOWSKI (1992): *Equivalences on observable processes*. In *Proc. LICS*, pp. 148–159. IEEE.
- [25] I. ULIDOWSKI & I. PHILLIPS (2002): *Ordered SOS rules and process languages for branching and eager bisimulations*. *Information and Computation*, 178:180–213, 2002.
- [26] I. ULIDOWSKI & S. YUEN (2000): *Process languages for rooted eager bisimulation*. In *Proc. CONCUR*, LNCS 1877, pp. 275–289. Springer.

A Modal Characterisations

We first prove the first part of Thm. 1, which states that \mathbb{O}_w is a modal characterisation of weak bisimilarity. We need to prove, given an LTS $(\mathbb{P}, \rightarrow)$, that $p \leftrightarrow_w q \Leftrightarrow p \sim_{\mathbb{O}_w} q$ for all $p, q \in \mathbb{P}$.

Proof: (\Rightarrow) Suppose $p \leftrightarrow_w q$, and $p \models \varphi$ for some $\varphi \in \mathbb{O}_w$. We prove $q \models \varphi$, by structural induction on φ . The reverse implication ($q \models \varphi$ implies $p \models \varphi$) follows by symmetry.

- $\varphi = \bigwedge_{i \in I} \varphi_i$. Then $p \models \varphi_i$ for $i \in I$. By induction $q \models \varphi_i$ for $i \in I$, so $q \models \bigwedge_{i \in I} \varphi_i$.
- $\varphi = \neg \varphi'$. Then $p \not\models \varphi'$. By induction $q \not\models \varphi'$, so $q \models \neg \varphi'$.
- $\varphi = \langle \varepsilon \rangle \varphi'$. Then $p \xrightarrow{\varepsilon} p' \models \varphi'$ for some term p' . Since $p \leftrightarrow_w q$, according to Def. 1, $q \xrightarrow{\varepsilon} q'$ for some q' with $p' \leftrightarrow_w q'$. Since $p' \models \varphi'$, by induction, $q' \models \varphi'$. Hence $q \models \langle \varepsilon \rangle \varphi'$.
- $\varphi = \langle \varepsilon \rangle \langle a \rangle \langle \varepsilon \rangle \varphi'$, with $a \in A$. Then $p \xrightarrow{\varepsilon} \xrightarrow{a} \xrightarrow{\varepsilon} p' \models \varphi'$ for some term p' . Since $p \leftrightarrow_w q$, according to Def. 1, $q \xrightarrow{\varepsilon} \xrightarrow{a} \xrightarrow{\varepsilon} q'$ for some q' with $p' \leftrightarrow_w q'$. Since $p' \models \varphi'$, by induction, $q' \models \varphi'$. Hence $q \models \langle \varepsilon \rangle \langle a \rangle \langle \varepsilon \rangle \varphi'$.

We conclude that $p \sim_{\mathbb{O}_w} q$.

(\Leftarrow) We prove that $\sim_{\mathbb{O}_w}$ is a weak bisimulation. The relation is clearly symmetric. Let $p \sim_{\mathbb{O}_w} q$. Suppose $p \xrightarrow{\alpha} p'$. If $\alpha = \tau$ and $p' \sim_{\mathbb{O}_w} q$, then the first condition of Def. 1 is fulfilled. So we can assume that either (i) $\alpha \neq \tau$ or (ii) $p' \not\sim_{\mathbb{O}_w} q$. Let

$$Q' := \{q' \in \mathbb{P} \mid q \xrightarrow{\varepsilon} \xrightarrow{\alpha} \xrightarrow{\varepsilon} q' \wedge p' \not\sim_{\mathbb{O}_w} q'\}.$$

For each $q' \in Q'$, let $\psi_{q'}$ be a formula in \mathbb{O}_w such that $p' \models \psi_{q'}$ and $q' \not\models \psi_{q'}$. We define

$$\psi = \bigwedge_{q' \in Q'} \psi_{q'}.$$

Clearly, $\psi \in \mathbb{O}_w$ and $p' \models \psi$. Moreover, $q' \not\models \psi$ for no $q' \in Q'$. We distinguish two cases.

1. $\alpha \neq \tau$. Since $p \models \langle \varepsilon \rangle \langle \alpha \rangle \langle \varepsilon \rangle \psi \in \mathbb{O}_w$ and $p \sim_{\mathbb{O}_w} q$, also $q \models \langle \varepsilon \rangle \langle \alpha \rangle \langle \varepsilon \rangle \psi$. Hence $q \xrightarrow{\varepsilon} \xrightarrow{\alpha} \xrightarrow{\varepsilon} q'$ with $q' \models \psi$. It follows that $q' \notin Q'$ and thus $p' \sim_{\mathbb{O}_w} q'$.
2. $\alpha = \tau$ and $p' \not\sim_{\mathbb{O}_w} q$. Let $\tilde{\varphi} \in \mathbb{O}_w$ such that $p' \models \tilde{\varphi}$ and $p, q \not\models \tilde{\varphi}$. Since $p \models \langle \varepsilon \rangle (\tilde{\varphi} \wedge \psi) \in \mathbb{O}_w$ and $p \sim_{\mathbb{O}_w} q$, also $q \models \langle \varepsilon \rangle (\tilde{\varphi} \wedge \psi)$. So $q \xrightarrow{\varepsilon} q'$ with $q' \models \tilde{\varphi} \wedge \psi$. It follows that $q' \notin Q'$. As $q \not\models \tilde{\varphi}$ we have $q' \neq q$ and thus $q \xrightarrow{\varepsilon} \xrightarrow{\tau} \xrightarrow{\varepsilon} q'$. Hence $p' \sim_{\mathbb{O}_w} q'$.

Both cases imply that the second condition of Def. 1 is fulfilled. We therefore conclude that $\sim_{\mathbb{O}_w}$ is a weak bisimulation. \square

Using the first part of Thm. 1, which was proved above, it is not hard to derive the second part of Thm. 1, i.e. that \mathbb{O}_{rw} is a modal characterisation of rooted weak bisimilarity.

Proof: (\Rightarrow) Suppose $p \leftrightarrow_{rw} q$, and $p \models \varphi$ for some $\varphi \in \mathbb{O}_{rw}$. We prove $q \models \varphi$, by structural induction on φ . The reverse implication ($q \models \varphi$ implies $p \models \varphi$) follows by symmetry.

- The cases $\varphi = \bigwedge_{i \in I} \varphi_i$ and $\varphi = \neg \varphi'$ go exactly as in the previous proof.
- $\varphi = \langle \varepsilon \rangle \langle \alpha \rangle \langle \varepsilon \rangle \hat{\varphi}$, with $\hat{\varphi} \in \mathbb{O}_w$. Then $p \xrightarrow{\varepsilon} \xrightarrow{\alpha} \xrightarrow{\varepsilon} p' \models \hat{\varphi}$ for some term p' . Since $p \leftrightarrow_{rw} q$, according to Def. 2, $q \xrightarrow{\varepsilon} \xrightarrow{\alpha} \xrightarrow{\varepsilon} q'$ for some q' with $p' \leftrightarrow_w q'$. Since $p' \models \hat{\varphi}$, by the previous result, $q' \models \hat{\varphi}$. Hence $q \models \langle \varepsilon \rangle \langle \alpha \rangle \langle \varepsilon \rangle \hat{\varphi}$.

- $\varphi \in \mathbb{O}_w$. Since $p \stackrel{\alpha}{\leftrightarrow}_{rw} q$ implies $p \stackrel{\alpha}{\leftrightarrow}_w q$, the previously result yields $q \models \varphi$.

We conclude that $p \sim_{\mathbb{O}_{rw}} q$.

(\Leftarrow) We prove that $\sim_{\mathbb{O}_{rw}}$ is a rooted weak bisimulation. The relation is clearly symmetric. Let $p \sim_{\mathbb{O}_{rw}} q$. Suppose $p \xrightarrow{\alpha} p'$. Let

$$Q' := \{q' \in \mathbb{P} \mid q \xrightarrow{\varepsilon} \xrightarrow{\alpha} \xrightarrow{\varepsilon} q' \wedge p' \not\sim_{\mathbb{O}_w} q'\}.$$

For each $q' \in Q'$, let $\psi_{q'}$ be a formula in \mathbb{O}_w such that $p' \models \psi_{q'}$ and $q' \not\models \psi_{q'}$. We define

$$\psi = \bigwedge_{q' \in Q'} \psi_{q'}.$$

Clearly, $\psi \in \mathbb{O}_w$ and $p' \models \psi$. Moreover, $q' \models \psi$ for no $q' \in Q'$.

Since $p \models \langle \varepsilon \rangle \langle \alpha \rangle \langle \varepsilon \rangle \psi \in \mathbb{O}_{rw}$ and $p \sim_{\mathbb{O}_{rw}} q$, also $q \models \langle \varepsilon \rangle \langle \alpha \rangle \langle \varepsilon \rangle \psi$. Hence $q \xrightarrow{\varepsilon} \xrightarrow{\alpha} \xrightarrow{\varepsilon} q'$ with $q' \models \psi$. It follows that $q' \notin Q'$ and thus $p' \sim_{\mathbb{O}_w} q'$. By the previous result this implies $p' \stackrel{\alpha}{\leftrightarrow}_w q'$.

Hence the condition of Def. 2 is fulfilled. We therefore conclude that $\sim_{\mathbb{O}_{rw}}$ is a rooted weak bisimulation. \square

The validity of the modal characterisation of (rooted) delay bisimilarity can be proved in a similar fashion.

B Manifest delay resistance

This appendix contains the proofs of Thms. 8 and 9.

B.1 τ -Pollable rules

As a major step towards Thms. 8 and 9, we would like to show that if all rules in a TSS P are delay resistant w.r.t. $[\Lambda \text{ and}] P$, then so are all ntytt rules that are linearly provable from P . However, we can prove this only if we assume all rules in P to have an additional property, which we call τ -pollability w.r.t. $[\Lambda \text{ and}] P$. In Sec. 3 we introduced the concept of a τ -pollable premise in a rule r : a positive premise that could be replaced by a similar premise with label τ and a fresh right-hand side. Below, a rule will be called τ -pollable if *all* its premises [with the exception of Λ -liquid ones] can be replaced in this manner; however, for standard rules this possibly comes at the expense of changing the label of the conclusion of r to τ , and its target to some term w , possibly containing the fresh right-hand sides mentioned above. It turns out that in a manifestly delay resistant TSS all rules are τ -pollable.

Definition 36 An ntytt rule $\frac{H}{t \xrightarrow{\alpha} \gamma}$, resp. $\frac{H}{t \xrightarrow{\alpha} u}$, is τ -pollable w.r.t. [a predicate $\Lambda \text{ and}]$ a TSS P if for each set $M \subseteq H^+[\setminus H^\Lambda]$ there is a rule $\frac{H_M}{t \xrightarrow{\alpha} \gamma}$, resp. $\frac{H_M}{t \xrightarrow{\alpha} u}$ or $\frac{H_M}{t \xrightarrow{\tau} w}$ for some term w , linearly provable from P , where $H_M \subseteq (H \setminus M) \cup M_\tau$.

We recall that M_τ was defined in Def 17; it is obtained from M by replacing the transition labels by τ , and the right-hand sides by fresh variables, not occurring in $\frac{H}{t \xrightarrow{\alpha} \gamma}$, resp. $\frac{H}{t \xrightarrow{\alpha} u}$. Those variables may occur in w , however.

Again, we introduce a “manifest” version of this concept, where the required rule needs to be in \bar{R} , rather than merely being linearly provable from P ; we consider this notion only for a TSS as a whole.

Definition 37 A TSS $P = (\Sigma, R)$ in ntytt format is called *manifestly τ -pollable* [w.r.t. Λ] if for each rule $\frac{H}{t \xrightarrow{\alpha} \gamma}$ or $\frac{H}{t \xrightarrow{\alpha} u}$ in R , and for each set $M \subseteq H^+[\backslash H^\Lambda]$, \bar{R} also contains a rule $\frac{H_M}{t \xrightarrow{\alpha} \gamma}$, resp. $\frac{H_M}{t \xrightarrow{\alpha} u}$ or $\frac{H_M}{t \xrightarrow{\tau} w}$ for some w , where $H_M \subseteq (H \setminus M) \cup M_\tau$.

Clearly, in a manifestly τ -pollable TSS P , each rule is τ -pollable w.r.t. P .

Lemma 10 Any manifestly delay resistant [w.r.t. Λ] standard TSS $P = (\Sigma, R)$ in decent ntyft format is manifestly τ -pollable [w.r.t. Λ].

Proof: Consider a rule $\frac{H}{t \xrightarrow{\alpha} u}$ in R and an $M \subseteq H^+[\backslash H^\Lambda]$. We apply induction on $|M|$, taking into account that M may be infinite. The induction base $M = \emptyset$ is trivial. In the induction step, $M \neq \emptyset$. Pick a finite set H^d of manifestly delayable positive premises with the property formulated in Def. 31. First we deal with the case that $M \not\subseteq H^d$. Let $M_1 := M \setminus H^d$. By Def. 31 there is a rule $\frac{H_1}{t \xrightarrow{\alpha} u}$ in \bar{R} with $H_1 \subseteq (H \setminus M_1) \cup (M_1)_\tau$. Let $M_2 := M \cap H^d \cap H_1^+$. As H^d is finite, so is M_2 . Since $M_2 \subseteq H_1^+[\backslash H_1^\Lambda]$ and $|M_2| < |M|$, by induction \bar{R} contains a rule $\frac{H_2}{t \xrightarrow{\alpha} u}$ or $\frac{H_2}{t \xrightarrow{\tau} w}$ for some term w , where $H_2 \subseteq (H_1 \setminus M_2) \cup (M_2)_\tau \subseteq (H \setminus M) \cup M_\tau$.

Next assume that $M \subseteq H^d$. As H^d is finite, so is M . Pick a $v \xrightarrow{\beta} y$ in M . Since it is a manifestly delayable premise of r , there exists a rule $\frac{H_1}{t \xrightarrow{\tau} w}$ in \bar{R} , for some term w and fresh variable z , with $H_1 \subseteq (H \setminus \{v \xrightarrow{\beta} y\}) \cup \{v \xrightarrow{\tau} z\}$. Let $M_1 := M \cap (H_1^+ \setminus \{v \xrightarrow{\tau} z\})$. Since $M_1 \subseteq H_1^+[\backslash H_1^\Lambda]$ and $|M_1| < |M|$, by induction \bar{R} contains a rule $\frac{H_2}{t \xrightarrow{\tau} w}$ for some term w , where $H_2 \subseteq (H_1 \setminus M_1) \cup (M_1)_\tau \subseteq (H \setminus M) \cup M_\tau$. (Here we use that there is a $v \xrightarrow{\tau} z_y$ in M_τ , and that we can choose $z_y := z$.) \square

B.2 Lifting manifest delay resistance from P to \hat{P}^\ddagger

We extend the predicate “positive delay resistant” to non-standard rules by declaring it vacuously true.

Lemma 11 Let P be a TSS in decent ntytt format, in which each transition rule is positive delay resistant as well as τ -pollable w.r.t. P . Then any ntytt rule linearly provable from P is positive delay resistant as well as τ -pollable w.r.t. P .

Proof: Let an ntytt rule $r = \frac{H}{t \xrightarrow{\alpha} \gamma}$ [resp. $\frac{H}{t \xrightarrow{\alpha} u}$] be linearly provable from P , by means of a proof π . We will prove, by structural induction on π , that this rule is positive delay resistant as well as τ -pollable w.r.t. P .

Induction basis: Suppose π has only one node. Then rule r has the form $\frac{t \xrightarrow{\alpha} \gamma}{t \xrightarrow{\alpha} \gamma}$ [resp. $\frac{t \xrightarrow{\alpha} \gamma}{t \xrightarrow{\alpha} \gamma}$]. The non-standard rule is trivially τ -pollable, by lack of positive premises. The standard rule satisfies the requirements of Defs. 17 and 16 by taking $H^d = H_\emptyset := \{t \xrightarrow{\alpha} \gamma\}$, $H_1 := \{t \xrightarrow{\tau} z\}$, $v := z$ and $H_2 := \{z \xrightarrow{\alpha} \gamma\}$. Furthermore, the requirement of Def. 36 is satisfied through the rule $\frac{t \xrightarrow{\tau} \gamma}{t \xrightarrow{\tau} \gamma}$.

Induction step: Let $r' = \frac{K}{t' \xrightarrow{\alpha} \gamma}$ [resp. $\frac{K}{t' \xrightarrow{\alpha} u'}$] be the rule and σ the substitution used at the bottom of π —by assumption, r' is decent, ntytt and positive delay resistant as well as τ -pollable w.r.t. P . Then $\sigma(t') = t$ [and $\sigma(u') = u$]. Moreover, rules $r_\mu = \frac{H_\mu}{\sigma(\mu)}$ for each $\mu \in K$ are linearly provable from P by means of strict subproofs of π , where $H = \bigcup_{\mu \in K} H_\mu$, and the sets H_μ are pairwise disjoint.

For each $\mu \in K$, let t_μ be the left-hand side of μ . As r' is decent, $\text{var}(t_\mu) \subseteq \text{var}(t')$, so $\text{var}(\sigma(t_\mu)) \subseteq \text{var}(\sigma(t')) = \text{var}(t)$. From $\text{rhs}(H) \cap \text{var}(t) = \emptyset$ it follows that $\text{rhs}(H_\mu) \cap \text{var}(\sigma(t_\mu)) = \emptyset$. So r_μ is an ntytt rule. By induction, r_μ is positive delay resistant as well as τ -pollable w.r.t. P .

To show that r is τ -pollable, pick any set $M \subseteq H^+$. It can be written as $M = \bigcup_{\mu \in K} M_\mu$ with $M_\mu \subseteq H_\mu^+$ for all $\mu \in K$. For each negative premise $\mu \in K$, as r_μ is τ -pollable, a rule $\frac{H_\mu^M}{\sigma(\mu)}$, with $H_\mu^M \subseteq (H_\mu \setminus M_\mu) \cup (M_\mu)_\tau$, is linearly provable from P . For each positive $\mu \in K$, say of the form $t_\mu \xrightarrow{\gamma_\mu} y_\mu$, as r_μ is τ -pollable, a rule $\frac{H_\mu^M}{\sigma(t_\mu) \xrightarrow{\gamma_\mu} \sigma(y_\mu)}$ or $\frac{H_\mu^M}{\sigma(t_\mu) \xrightarrow{\tau} w_\mu}$, with $H_\mu^M \subseteq (H_\mu \setminus M_\mu) \cup (M_\mu)_\tau$, is linearly provable from P . In the construction of the sets $(M_\mu)_\tau$ (see Def. 17), we make sure that the fresh right-hand sides z_y are all different. Let M^K be the set of literals μ from K^+ for which only the second of these two possibilities applies. Since r' is τ -pollable, there is a rule $\frac{K^M}{t' \xrightarrow{\alpha} u'}$ [resp. $\frac{K^M}{t' \xrightarrow{\alpha} u'}$ or $\frac{K^M}{t' \xrightarrow{\tau} w'}$], linearly provable from P , with $K^M \subseteq (K \setminus M^K) \cup M^K_\tau$. Lem. 2 yields a linear proof from P , which uses this rule and σ at the bottom, of a rule $\frac{H^M}{t \xrightarrow{\alpha} u}$ [resp. $\frac{H^M}{t \xrightarrow{\alpha} u}$ or $\frac{H^M}{t \xrightarrow{\tau} w}$] with $H^M \subseteq \bigcup_{\mu \in K} H_\mu^M = (H \setminus M) \cup M_\tau$, as required by Def. 36.

It remains to show that r is positive delay resistant. So assume $r = \frac{H}{t \xrightarrow{\alpha} u}$ and $r' = \frac{K}{t' \xrightarrow{\alpha} u'}$. Let $K^d \subseteq K^+$ and $H_\mu^d \subseteq H_\mu^+$ for each $\mu \in K$ be the finite sets of delayable positive premises of r' resp. r_μ , which exist by Def. 17. Take $H^d := \bigcup_{\mu \in K^d} H_\mu^d$. Pick any set $M \subseteq H^+ \setminus H^d$. It can be written as $M = \bigcup_{\mu \in K} M_\mu$ with $M_\mu \subseteq H_\mu^+$ for each $\mu \in K$; moreover, $M_\mu \subseteq H_\mu^+ \setminus H_\mu^d$ for each $\mu \in K^d$. For each negative premise $\mu \in K$, as r_μ is τ -pollable, a rule $\frac{H_\mu^M}{\sigma(\mu)}$ with $H_\mu^M \subseteq (H_\mu \setminus M_\mu) \cup (M_\mu)_\tau$ is linearly provable from P . For each $\mu \in K^+$, say of the form $t_\mu \xrightarrow{\gamma_\mu} y_\mu$, as r_μ is τ -pollable, a rule $\frac{H_\mu^M}{\sigma(t_\mu) \xrightarrow{\gamma_\mu} \sigma(y_\mu)}$ or $\frac{H_\mu^M}{\sigma(t_\mu) \xrightarrow{\tau} w_\mu}$ with $H_\mu^M \subseteq (H_\mu \setminus M_\mu) \cup (M_\mu)_\tau$ is linearly provable from P . In the construction of the sets $(M_\mu)_\tau$ we make sure that the fresh right-hand sides z_y are all different. In the special case that $\mu \in K^d$ we have $M_\mu \subseteq H_\mu^+ \setminus H_\mu^d$, so that the positive delay resistance of r_μ guarantees that the first of these two possibilities applies. Let M^K be the set of $\mu \in K^+$ for which only the second possibility applies. Then $M^K \subseteq K^+ \setminus K^d$. Since r' is positive delay resistant, there is a rule $\frac{K^M}{t' \xrightarrow{\alpha} u'}$, linearly provable from P , with $K^M \subseteq (K \setminus M^K) \cup M^K_\tau$. Lem. 2 yields a linear proof from P , which uses this rule and σ at the bottom, of a rule $\frac{H^M}{t \xrightarrow{\alpha} u}$ with $H^M \subseteq \bigcup_{\mu \in K} H_\mu^M = (H \setminus M) \cup M_\tau$, as required by Def. 17.

It remains to show that any literal in H^d , say of the form $w \xrightarrow{\beta} y$, is a delayable premise of r . By the definition of H^d , $w \xrightarrow{\beta} y$ is in $H_{\mu_0}^d$ for some $\mu_0 \in K^d$; say μ_0 is of the form $t_0 \xrightarrow{\gamma} y_0$. Since $w \xrightarrow{\beta} y$ is a delayable premise of r_{μ_0} , there are rules $\frac{H_{\mu_0}^1}{\sigma(t_0) \xrightarrow{\tau} v}$ and $\frac{H_{\mu_0}^2}{v \xrightarrow{\gamma} \sigma(y_0)}$, linearly provable from P , with $H_{\mu_0}^1 \subseteq (H_{\mu_0} \setminus \{w \xrightarrow{\beta} y\}) \cup \{w \xrightarrow{\tau} z\}$ and $H_{\mu_0}^2 \subseteq (H_{\mu_0} \setminus \{w \xrightarrow{\beta} y\}) \cup \{z \xrightarrow{\beta} y\}$ for some term v and fresh variable z . Likewise, since $t_0 \xrightarrow{\gamma} y_0$ is a delayable premise of r' , there are rules $\frac{K_1}{t' \xrightarrow{\tau} v'}$ and $\frac{K_2}{v' \xrightarrow{\alpha} u'}$, linearly provable from P , with $K_1 \subseteq (K \setminus \{t_0 \xrightarrow{\gamma} y_0\}) \cup \{t_0 \xrightarrow{\tau} z'\}$ and $K_2 \subseteq (K \setminus \{t_0 \xrightarrow{\gamma} y_0\}) \cup \{z' \xrightarrow{\gamma} y_0\}$ for some term v' and fresh variable z' . Without limitation of generality, we pick z' so that it does not occur in π . Let $\sigma'(z') = v$ and σ' coincides with σ on all other variables. Let L_1 and L_2 denote $\bigcup_{\mu \in K_1 \setminus \{t_0 \xrightarrow{\tau} z'\}} H_\mu \cup H_{\mu_0}^1$ resp. $\bigcup_{\mu \in K_2 \setminus \{z' \xrightarrow{\gamma} y_0\}} H_\mu \cup H_{\mu_0}^2$. Lem. 2 yields linear proofs from P , which use $\frac{K_1}{t' \xrightarrow{\tau} v'}$ and $\frac{K_2}{v' \xrightarrow{\alpha} u'}$ and σ' at the bottom, of the rules $\frac{L_1}{t \xrightarrow{\tau} \sigma'(v')}$ and $\frac{L_2}{\sigma'(v') \xrightarrow{\alpha} u}$. Moreover, $L_1 \subseteq (H \setminus \{w \xrightarrow{\beta} y\}) \cup \{w \xrightarrow{\tau} z\}$ and $L_2 \subseteq (H \setminus \{w \xrightarrow{\beta} y\}) \cup \{z \xrightarrow{\beta} y\}$. \square

Only in the above proof, together with its forthcoming variant proving Lem. 14, does it make a difference that linear provability is used instead of irredundant provability. It allows us to infer that the sets H_μ are pairwise disjoint. Without that, the last sentence in the proof would fail. See also Ex. 8.

Lemma 12 Let P be a negative-stable standard TSS in decent ntytt format. Then any ntytt rule irredundantly provable from P is negative-stable.

Proof: Let an ntytt rule $r = \frac{H}{t \xrightarrow{\alpha} u}$ be irredundantly provable from P , by means of a proof π . We will prove, by structural induction on π , that r is negative-stable.

Induction basis: If π has only one node, then $\frac{H}{t \xrightarrow{\alpha} u}$ equals $\frac{t \xrightarrow{\alpha} u}{t \xrightarrow{\alpha} u}$, which trivially is negative-stable.

Induction step: Let $r' = \frac{K}{t' \xrightarrow{\alpha} u'}$ be the rule and σ the substitution used at the bottom of π —by assumption, r' is decent, ntytt and negative-stable; so $K = K^+ \cup K^{s-}$. Then $\sigma(t') = t$ and $\sigma(u') = u$. Moreover, rules $r_\mu = \frac{H_\mu}{\sigma(\mu)}$ for each $\mu \in K^+$ are irredundantly provable from P by means of strict subproofs of π , where $H = \bigcup_{\mu \in K^+} H_\mu \cup \sigma(K^{s-})$.

For each $\mu \in K^+$, let t_μ be the left-hand side of μ . As r' is decent, $\text{var}(t_\mu) \subseteq \text{var}(t')$, so $\text{var}(\sigma(t_\mu)) \subseteq \text{var}(\sigma(t')) = \text{var}(t)$. From $\text{rhs}(H) \cap \text{var}(t) = \emptyset$ it follows that $\text{rhs}(H_\mu) \cap \text{var}(\sigma(t_\mu)) = \emptyset$. So r_μ is an ntytt rule. By induction, r_μ is negative-stable. From this it follows that r is negative-stable. \square

Proposition 11 Let P be a standard TSS in ready simulation format. If P is manifestly delay resistant, then so is the standard TSS \hat{P}^\ddagger in xynft format constructed in Secs. 2.6–2.7.

Proof: Let P be manifestly delay resistant. By Def. 32, the conversion P^\dagger of P to decent ntyft format, defined in Sec. 2.6, is manifestly delay resistant. By Lem. 10 it is also manifestly τ -pollable. Hence all its rules are τ -pollable w.r.t. P^\dagger , and manifestly delay resistant w.r.t. P^\dagger —thus negative-stable and positive delay resistant w.r.t. P^\dagger .

The TSS \hat{P}^\ddagger contains all xynft rules linearly provable from P^\dagger . By Lem. 12 those rules are negative-stable. By Lem. 11 they are positive delay resistant and τ -pollable w.r.t. P^\dagger . The definitions of positive delay resistant and τ -pollability of rules from \hat{P}^\ddagger w.r.t. P^\dagger require the existence of certain xynft rules that are linearly provable from P^\dagger . By definition, these are rules of \hat{P}^\ddagger . Hence \hat{P}^\ddagger is manifestly delay resistant, as well as manifestly τ -pollable. \square

B.3 Eliminating the Λ -restriction

Prop. 11 above is a crucial step (or “halfway marker”) in the proof of Thm. 8. In this section we take a similar step (Prop. 12) towards the proof of Thm. 9. It requires P to be merely manifestly delay resistant w.r.t. Λ , rather than outright manifestly delay resistant. On the other hand, it assumes the extra antecedents of Thm. 9. The conclusion of Prop. 12 is that P is manifestly delay resistant. This allows us to dispense with Λ in the second half of the proof of Thm. 9, which therefore will equal the second half of the proof of Thm. 8.

Our initial proof strategy was to extend Lem. 11—similar to Lem. 10—by replacing “w.r.t. P ” by “w.r.t. $[\Lambda \text{ and }] P$ ” both in the antecedent and in the conclusion. Then Lemmas 7–9 would suffice to establish Prop. 12, eliminating Λ . This strategy failed, however. It turns out we have to integrate the proof of Lem. 7 into the proof of the modified Lem. 11, and eliminate Λ while lifting positive delay resistance and τ -pollability from the rules of P to the linearly provable xynft rules. This yields the forthcoming Lem. 14.

The following lemma is a variant of Prop. 1, needed in the proof of Lem. 14. We write $P \vdash_{\text{lin}} r$ to say that a rule r is linearly provable from a TSS P .

Lemma 13 Let $P = (\Sigma, R)$ be a TSS in decent ntyft format, $t, t' \in \mathbb{T}(\Sigma)$ and σ a substitution. If $P \vdash_{\text{lin}} r$ with $r = \frac{H}{\sigma(t) \xrightarrow{\alpha} t}$ [resp. $\frac{H}{\sigma(t) \xrightarrow{\alpha} t'}$] an xyntt rule, then there are a decent xyntt rule $r' = \frac{G}{t \xrightarrow{\alpha} u}$ [resp. $\frac{G}{t \xrightarrow{\alpha} u}$] with $P \vdash_{\text{lin}} r'$ and a substitution σ' with $\sigma'(t) = \sigma(t)$ [and $\sigma'(u) = t'$] such that H can be written as $\biguplus_{v \in G} H_v$ with $P \vdash_{\text{lin}} \frac{H_v}{\sigma'(v)}$ for all $v \in G$. Moreover, any proof π of r from P has subproofs π_v of $\frac{H_v}{\sigma'(v)}$; and if neither t is a variable nor π a 1-node proof, the π_v are strict subproofs of π .

Proof: First, suppose t is a variable. By default, the decent xyntt rule $\frac{t \xrightarrow{\alpha} y}{t \xrightarrow{\alpha} y}$ [resp. $\frac{t \xrightarrow{\alpha} y}{t \xrightarrow{\alpha} y}$] is linearly provable from P . Let σ' be a substitution with $\sigma'(t) = \sigma(t)$ [and $\sigma'(y) = t'$]. Clearly, $P \vdash_{\text{lin}} \frac{H}{\sigma'(t \xrightarrow{\alpha} y)}$ [resp. $P \vdash_{\text{lin}} \frac{H}{\sigma'(t \xrightarrow{\alpha} y)}$], taking $\pi_v := \pi$.

Next, suppose $t = f(t_1, \dots, t_{ar(f)})$. We apply structural induction on the proof π of r from P .

Induction basis: If π has only one node, using that r is an xyntt rule, $r = \frac{t \xrightarrow{\alpha} y}{t \xrightarrow{\alpha} y}$. Take $r' := r$, $\sigma' := \sigma$, $H_v := H$ and $\pi_v := \pi$.

Induction step: Let $r'' \in R$ be the decent ntyft rule and ρ the substitution used at the bottom of π , where r'' is of the form $\frac{\{v_k \xrightarrow{\gamma_k} y_k | k \in K\} \cup \{w_\ell \xrightarrow{\delta_\ell} y_\ell | \ell \in L\}}{f(x_1, \dots, x_{ar(f)}) \xrightarrow{\alpha} y}$ [resp. $\frac{\{v_k \xrightarrow{\gamma_k} y_k | k \in K\} \cup \{w_\ell \xrightarrow{\delta_\ell} y_\ell | \ell \in L\}}{f(x_1, \dots, x_{ar(f)}) \xrightarrow{\alpha} y}$]. Then $\rho(x_i) = \sigma(t_i)$ for $i = 1, \dots, ar(f)$, $[\rho(v) = t']$ and rules $\frac{H_k}{\rho(v_k) \xrightarrow{\gamma_k} \rho(y_k)}$ for $k \in K$ and $\frac{H_\ell}{\rho(w_\ell) \xrightarrow{\delta_\ell} \rho(y_\ell)}$ for $\ell \in L$ are linearly provable from P by means of strict subproofs of π , where $H = \biguplus_{k \in K} H_k \cup \biguplus_{\ell \in L} H_\ell$. Since r'' is decent, $var(v_k)$ for $k \in K$ and $var(w_\ell)$ for $\ell \in L$ are included in $\{x_1, \dots, x_{ar(f)}\}$. Let ρ_0 be a substitution with $\rho_0(x_i) = t_i$ for $i = 1, \dots, ar(f)$. As $\rho(x_i) = \sigma(t_i) = \sigma(\rho_0(x_i))$ for $i = 1, \dots, ar(f)$, we have $\rho(v_k) = \sigma(\rho_0(v_k))$ for $k \in K$ and $\rho(w_\ell) = \sigma(\rho_0(w_\ell))$ for $\ell \in L$. So $\frac{H_k}{\sigma(\rho_0(v_k)) \xrightarrow{\gamma_k} \rho(y_k)}$ for $k \in K$ and $\frac{H_\ell}{\sigma(\rho_0(w_\ell)) \xrightarrow{\delta_\ell} \rho(y_\ell)}$ are linearly provable from P by means of strict subproofs π_k and π_ℓ of π . According to the induction hypothesis, for $k \in K$ there are a decent xyntt rule $\frac{G_k}{\rho_0(v_k) \xrightarrow{\gamma_k} u_k}$ and a substitution σ'_k with $P \vdash_{\text{lin}} \frac{G_k}{\rho_0(v_k) \xrightarrow{\gamma_k} u_k}$, $\sigma'_k(\rho_0(v_k)) = \sigma(\rho_0(v_k))$ and $\sigma'_k(u_k) = \rho(y_k)$, such that H_k can be written as $\biguplus_{v \in G_k} H_v$ with $P \vdash_{\text{lin}} \frac{H_v}{\sigma'_k(v)}$ for all $v \in G_k$. Moreover, π_k has subproofs π_v of $\frac{H_v}{\sigma'_k(v)}$. Likewise, for $\ell \in L$ there are a decent xyntt rule $\frac{G_\ell}{\rho_0(w_\ell) \xrightarrow{\delta_\ell} u_\ell}$ and a substitution σ'_ℓ with $P \vdash_{\text{lin}} \frac{G_\ell}{\rho_0(w_\ell) \xrightarrow{\delta_\ell} u_\ell}$ and $\sigma'_\ell(\rho_0(w_\ell)) = \sigma(\rho_0(w_\ell))$, such that H_ℓ can be written as $\biguplus_{v \in G_\ell} H_v$ with $P \vdash_{\text{lin}} \frac{H_v}{\sigma'_\ell(v)}$ for all $v \in G_\ell$. Moreover, π_ℓ has subproofs π_v of $\frac{H_v}{\sigma'_\ell(v)}$. As observed in [2], using that $|\Sigma|, |A| \leq |V|$, we can choose the sets of variables in the right-hand sides of the positive premises in the G_k (for $k \in K$) and G_ℓ (for $\ell \in L$) pairwise disjoint, and disjoint from $var(t)$. This allows us to define a substitution σ' with:

- $\sigma'(z) = \sigma(z)$ for $z \in var(t)$;
- $\sigma'(z) = \sigma'_k(z)$ for right-hand sides z of positive premises in G_k for $k \in K$;
- $\sigma'(z) = \sigma'_\ell(z)$ for right-hand sides z of positive premises in G_ℓ for $\ell \in L$.

Let G denote $\bigcup_{k \in K} G_k \cup \bigcup_{\ell \in L} G_\ell$. Moreover, let ρ_1 be a substitution with $\rho_1(x_i) = t_i$ for $i = 1, \dots, ar(f)$ and $\rho_1(y_k) = u_k$ for $k \in K$. We verify that the rule $\frac{G}{t \xrightarrow{\alpha} \rho_1(v)}$ [resp. $\frac{G}{t \xrightarrow{\alpha} \rho_1(v)}$] together with the substitution σ' satisfy the desired properties.

As $var(v_k) \subseteq \{x_1, \dots, x_{ar(f)}\}$, it follows that $var(\rho_0(v_k)) \subseteq var(t)$. Since σ' and σ agree on $var(t)$, $\sigma'(\rho_0(v_k)) = \sigma(\rho_0(v_k)) = \sigma'_k(\rho_0(v_k))$ for $k \in K$. Thus, by the decency of $\frac{G_k}{\rho_0(v_k) \xrightarrow{\gamma_k} u_k}$, σ' and σ'_k agree on all variables occurring in this rule for $k \in K$. Likewise, σ' and σ'_ℓ agree on all variables occurring in $\frac{G_\ell}{\rho_0(w_\ell) \xrightarrow{\delta_\ell} u_\ell}$ for $\ell \in L$.

1. ρ_1 and ρ_0 agree on $\text{var}(v_k) \subseteq \{x_1, \dots, x_{\text{ar}(f)}\}$, so $\rho_1(v_k) = \rho_0(v_k)$ for $k \in K$. Likewise, $\rho_1(w_\ell) = \rho_0(w_\ell)$ for $\ell \in L$. Since $P \vdash_{\text{lin}} r''$, we have $P \vdash_{\text{lin}} \rho_1(r'') = \frac{\{\rho_0(v_k) \xrightarrow{\gamma_k} u_k | k \in K\} \cup \{\rho_0(w_\ell) \xrightarrow{\delta_\ell} | \ell \in L\}}{t \xrightarrow{\alpha} \rho_1(v)}$ [resp. $\frac{\{\rho_0(v_k) \xrightarrow{\gamma_k} u_k | k \in K\} \cup \{\rho_0(w_\ell) \xrightarrow{\delta_\ell} | \ell \in L\}}{t \xrightarrow{\alpha} \rho_1(v)}$].
Furthermore, $P \vdash_{\text{lin}} \frac{G_k}{\rho_0(v_k) \xrightarrow{\gamma_k} u_k}$ for $k \in K$ and $P \vdash_{\text{lin}} \frac{G_\ell}{\rho_0(w_\ell) \xrightarrow{\delta_\ell} }$ for $\ell \in L$. As $G = \bigcup_{k \in K} G_k \cup \bigcup_{\ell \in L} G_\ell$, it follows that $P \vdash_{\text{lin}} \frac{G}{t \xrightarrow{\alpha} \rho_1(v)}$ [resp. $P \vdash_{\text{lin}} \frac{G}{t \xrightarrow{\alpha} \rho_1(v)}$].
2. The right-hand sides of the positive premises in any G_k or G_ℓ are distinct variables. By construction, these sets of variables (one for every $k \in K$ and $\ell \in L$) are pairwise disjoint, and disjoint from $\text{var}(t)$. Hence $\frac{G}{t \xrightarrow{\alpha} \rho_1(v)}$ [resp. $\frac{G}{t \xrightarrow{\alpha} \rho_1(v)}$] is an ntytt rule. Since the positive premises in G originate from G_k (for $k \in K$) and G_ℓ (for $\ell \in L$), their left-hand sides are variables. This makes the rule an xyntt rule. The rule is decent by Lem. 1.
3. Since σ' and σ agree on $\text{var}(t)$, $\sigma'(t) = \sigma(t)$.
4. $[\sigma'(\rho_1(x_i)) = \sigma'(t_i) = \sigma(t_i) = \rho(x_i)$ for $i = 1, \dots, \text{ar}(f)$. Moreover, since σ' and σ'_k agree on $\text{var}(u_k)$, $\sigma'(\rho_1(y_k)) = \sigma'(u_k) = \sigma'_k(u_k) = \rho(y_k)$ for $k \in K$. As $\text{var}(v) \subseteq \{x_1, \dots, x_{\text{ar}(f)}\} \cup \{y_k | k \in K\}$, it follows that $\sigma'(\rho_1(v)) = \rho(v) = t'$.]
5. $H = \bigsqcup_{k \in K} H_k \cup \bigsqcup_{\ell \in L} H_\ell = \bigsqcup_{k \in K} \bigsqcup_{v \in G_k} H_v \cup \bigsqcup_{\ell \in L} \bigsqcup_{v \in G_\ell} H_v = \bigsqcup_{v \in G} H_v$.
6. $P \vdash_{\text{lin}} \frac{H_v}{\sigma'(v)}$ for all $v \in G$, using that $\sigma'(v) = \sigma'_k(v)$ when $v \in G_k$ and $\sigma'(v) = \sigma'_\ell(v)$ when $v \in G_\ell$.
7. For $v \in G$ the proof π has strict subproofs π_v of $\frac{H_v}{\sigma'(v)}$. □

With this result in hand, we obtain the following variant of Lem. 11.

Lemma 14 Let P be an $\aleph \cap \Lambda$ -patient TSS in decent ntyft format, in which each transition rule is rooted branching bisimulation safe and satisfies condition 5 of Def. 27 w.r.t. \aleph and Λ , and is positive delay resistant as well as τ -pollable w.r.t. Λ and P . Then any xyntt rule linearly provable from P is positive delay resistant as well as τ -pollable w.r.t. P .

Note the absence of the disclaimer ‘‘w.r.t. Λ ’’ in the conclusion of the lemma.

Proof: This proof expands the proof of Lem. 11; we only explain the non-trivial differences. The first difference is that we apply structural induction on the *skeleton* of a proof π , rather than on π itself. Here the skeleton is the tree labelled with the rules that are applied in each node, but not with the substitutions used, or the resulting literals. This allows us to apply the induction hypothesis on proofs π' whose skeleton is a proper subtree of the skeleton of the proof π currently under investigation, even when π' itself is not a subtree of π .

In the proof of Lem. 11 we constructed, for each proof π of an (ntytt) rule $r = \frac{H}{t \xrightarrow{\alpha} u}$, a finite set $H^d \subseteq H^+$ of delayable premises of r . Here, we do this so that if π^\dagger is a strict subproof of π , proving an (xyntt) rule $r^\dagger = \frac{H^\dagger}{v}$ (so that $H^\dagger \subseteq H$) with $H^\dagger \cap H^d \neq \emptyset$, then v is positive and any $\mu \in H^\dagger \cap H^d$ is a delayable premise also of r^\dagger . (*)

When starting with an xyntt rule $r = \frac{H}{t \xrightarrow{\alpha} u}$ [resp. $\frac{H}{t \xrightarrow{\alpha} u}$], linearly provable from P by means of a proof π , we first deal with the case that t is univariate. The general case is dealt with at the end of this proof.

Following the proof of Lem. 11, in the induction step $r' = \frac{K}{t' \xrightarrow{\alpha'} u'}$ [resp. $\frac{K}{t' \xrightarrow{\alpha'} u'}$] is positive delay resistant and τ -pollable w.r.t. Λ and P , whereas the $r_\mu = \frac{H_\mu}{\sigma(\mu)}$ for $\mu \in K$ are positive delay resistant and τ -pollable

w.r.t. P ; they also satisfy (*) for subproofs of π . By Lem. 4, r is rooted branching bisimulation safe w.r.t. \aleph and Λ .

To show that r is τ -pollable, proceed as in the proof of Lem. 11, until the appearance of the set M^K . In case $M^K \subseteq K^+ \setminus K^\Lambda$ we proceed exactly as in the proof of Lem. 11, using that r' is τ -pollable w.r.t. Λ and P . So now assume that $\mu \in M^K \cap K^\Lambda$. Consider a premise $\lambda = (x \xrightarrow{\beta} y)$ in M_μ . By the decency of r (cf. Lem. 1), $x \in \text{var}(t)$. Using that t is univariate, there is exactly one variable $x_0 \in \text{var}(t')$ with $x \in \text{var}(\sigma(x_0))$. By the decency of r_μ (cf. Lem. 1), $x \in \text{var}(\sigma(t_\mu))$, and using the decency of r' this implies that $x_0 \in \text{var}(t_\mu)$. Given that $\mu \in K^\Lambda$, the occurrence of x_0 in t' must be Λ -liquid. By Lem. 13, the proof π_μ must have a subproof π' of a rule $\frac{H'}{\sigma(x_0) \xrightarrow{\gamma} u'}$ or $\frac{H'}{\sigma(x_0) \xrightarrow{\gamma} u''}$ with $(x \xrightarrow{\beta} y) \in H' \subseteq H_\mu$. By induction, this rule is τ -pollable w.r.t. P , so for $M' := M_\mu \cap H'$ there is a rule $\frac{H'_{M'}}{\sigma(x_0) \xrightarrow{\alpha} u'}$, resp. $\frac{H'_{M'}}{\sigma(x_0) \xrightarrow{\alpha} u''}$ or $\frac{H'_{M'}}{\sigma(x_0) \xrightarrow{\tau} w}$ for some term w , linearly provable from P , where $H'_{M'} \subseteq (H' \setminus M') \cup M'_\tau$. Call the premise λ *innocent* if in fact there is a rule $\frac{H'_{M'}}{\sigma(x_0) \xrightarrow{\alpha} u'}$, resp. $\frac{H'_{M'}}{\sigma(x_0) \xrightarrow{\alpha} u''}$. If all premises in M_μ are innocent, given the construction of rules required by Def. 36, there is no reason for μ to be in M^K . So there must be a guilty premise $\lambda = (x \xrightarrow{\beta} y)$ in M_μ such that a rule $\frac{H'_{M'}}{\sigma(x_0) \xrightarrow{\tau} w}$ for some term w is linearly provable from P . Since x occurs \aleph -liquid in a premise of H , its unique occurrence in t must also be \aleph -liquid, using condition 3 of Def. 21. This implies that the unique occurrence of x_0 in t' is $\aleph \cap \Lambda$ -liquid. Hence an $\aleph \cap \Lambda$ -patient rule $\frac{x_0 \xrightarrow{\tau} z_0}{t' \xrightarrow{\tau} w'}$, where w' is t' with x_0 replaced by z_0 , is linearly provable from P . Let $\sigma'(z_0) = w$ and σ' coincides with σ on all other variables. Lem. 2 yields a linear proof, which uses $\frac{x_0 \xrightarrow{\tau} z_0}{t' \xrightarrow{\tau} w'}$ and σ' at the bottom, of the rule $\frac{H'_{M'}}{t \xrightarrow{\tau} \sigma'(w')}$, where $H'_{M'} \subseteq (H' \setminus M') \cup M'_\tau \subseteq (H_\mu \setminus M_\mu) \cup (M_\mu)_\tau \subseteq (H \setminus M) \cup M_\tau$, as required.

To show that r is positive delay resistant, and satisfies (*), assume that r has the form $r = \frac{H}{t \xrightarrow{\alpha} u}$, and $r' = \frac{K}{t' \xrightarrow{\alpha} u'}$. As in the proof of Lem. 11, let $K^d \subseteq K$ and, for each $\mu \in K$, let $H_\mu^d \subseteq H_\mu^+$ be the finite sets of positive premises that exist by Def. 17. This time, take $H^d := \bigcup_{\mu \in K^d \cup (K^\Lambda)^+} H_\mu^d$. The requirement of Def. 17 is established exactly as in the proof of Lem. 11, but substituting $K^d \cup (K^\Lambda)^+$ for K^d . By construction (and induction) it follows that r satisfies (*) w.r.t. π .

It remains to show that any literal in H^d , say of the form $x \xrightarrow{\beta} y$, is a delayable premise of r . There is a μ_0 of the form $t_0 \xrightarrow{\gamma} y_0$ in $K^d \cup K^\Lambda$ with $x \xrightarrow{\beta} y$ in $H_{\mu_0}^d$. The case that $\mu_0 \in K^d$ proceeds exactly as in the proof of Lem. 11. So let $\mu_0 \in K^\Lambda$. By the decency of r (cf. Lem. 1), $x \in \text{var}(t)$. Using that t is univariate, there is exactly one variable $x_0 \in \text{var}(t')$ with $x \in \text{var}(\sigma(x_0))$. By the decency of r_{μ_0} (cf. Lem. 1), $x \in \text{var}(\sigma(t_{\mu_0}))$, and using the decency of r' this implies that $x_0 \in \text{var}(t_{\mu_0})$. Given that $\mu_0 \in K^\Lambda$, the occurrence of x_0 in t' must be Λ -liquid.

By Lem. 13 there is a decent xyntt rule $r'' = \frac{G}{t' \xrightarrow{\alpha} u''}$ with $P \vdash_{\text{lin}} r''$, and a substitution σ' with $\sigma'(t') = \sigma(t') = t$ and $\sigma'(u'') = \sigma(u'') = u$, such that H can be written as $\bigvee_{v \in G} H_v$ with $P \vdash_{\text{lin}} \frac{H_v}{\sigma'(v)}$ for all $v \in G$. Moreover, proofs π_v of $\frac{H_v}{\sigma'(v)}$ from P occur as strict subproofs of π . Let v_0 be the unique literal in G be such that $x \xrightarrow{\beta} y$ occurs in H_{v_0} . By (*), v_0 is positive—so has the form $x_0 \xrightarrow{\delta} y'_0$ —and $x \xrightarrow{\beta} y$ is a delayable premise of $\frac{H_{v_0}}{\sigma'(v_0)}$. Hence, there are rules $\frac{H_{v_0}^1}{\sigma'(x_0) \xrightarrow{\tau} v}$ and $\frac{H_{v_0}^2}{v \xrightarrow{\delta} \sigma'(y'_0)}$, linearly provable from P , with $H_{v_0}^1 \subseteq (H_{v_0} \setminus \{x \xrightarrow{\beta} y\}) \cup \{x \xrightarrow{\tau} z\}$ and $H_{v_0}^2 \subseteq (H_{v_0} \setminus \{x \xrightarrow{\beta} y\}) \cup \{z \xrightarrow{\beta} y\}$ for some term v and fresh variable z .

Since x occurs \aleph -liquid in a premise of H , its unique occurrence in t must also be \aleph -liquid, using

condition 3 of Def. 21. This implies that the unique occurrence of x_0 in t' is $\aleph \cap \Lambda$ -liquid. Hence an $\aleph \cap \Lambda$ -patient rule $\frac{x_0 \xrightarrow{\tau} z_0}{t' \xrightarrow{\tau} w'}$, where w' is t' with x_0 replaced by z_0 , is linearly provable from P . Let $\sigma''(z_0) = v$ and σ'' coincides with σ on all other variables. As z_0 is chosen fresh, $\sigma''(x_0) = \sigma'(x_0)$, $\sigma''(t') = \sigma'(t') = t$ and $\sigma''(u') = \sigma'(u') = u$. Lem. 2 yields a linear proof, which uses $\frac{x_0 \xrightarrow{\tau} z_0}{t' \xrightarrow{\tau} w'}$ and σ'' at the bottom, of the rule $\frac{H_{v_0}^1}{t \xrightarrow{\tau} \sigma''(w')}$, with $H_{v_0}^1 \subseteq (H_{v_0} \setminus \{x \xrightarrow{\beta} y\}) \cup \{x \xrightarrow{\tau} z\} \subseteq H \setminus \{x \xrightarrow{\beta} y\} \cup \{x \xrightarrow{\tau} z\}$.

By Lem. 6, the rule r'' satisfies condition 5 of Def. 27 w.r.t. \aleph and Λ . So the occurrences of x_0 in t' and in v_0 are the only two occurrences of x_0 in r'' . Replacing these occurrences of x_0 in r'' by z_0 produces a rule r''' with source w' and a premise $z_0 \xrightarrow{\delta} y'_0$. Let L denote $\bigcup_{v \in G \setminus \{v_0\}} H_v \cup H_{v_0}^2$. Lem. 2 yields a linear proof, which uses r''' and σ'' at the bottom, of the rule $\frac{L}{\sigma''(w') \xrightarrow{\alpha} u}$, with $L \subseteq (H \setminus \{x \xrightarrow{\beta} y\}) \cup \{x \xrightarrow{\tau} z\}$.

We have now finished the case that t is univariate. Suppose an xyntt rule $r^\ddagger = \frac{H^\ddagger}{t^\ddagger \xrightarrow{\alpha} u^\ddagger}$ [resp. $\frac{H^\ddagger}{t^\ddagger \xrightarrow{\alpha} u^\ddagger}$], with t^\ddagger not univariate, is linearly provable from P by means of a proof π^\ddagger . By Lem. 8 there is an xyntt rule $r = \frac{H}{t \xrightarrow{\alpha} u}$ [resp. $\frac{H}{t \xrightarrow{\alpha} u}$] with t univariate, such that $r^\ddagger = \rho(r)$, using a substitution $\rho : \text{dom}(t) \rightarrow V$. From the (trivial) proof of Lem. 8 we learn that r has a proof π that has the same skeleton as π^\ddagger ; in fact, $\pi^\ddagger = \rho(\pi)$. Hence, the induction hypothesis applies to strict subproofs of π just as much as to strict subproofs of π^\ddagger .

We have shown above that r is positive delay resistant as well as τ -pollable w.r.t. P , and also satisfies (*) w.r.t. the proof π . Now Lem. 9 says that also r^\ddagger is positive delay resistant w.r.t. P . In the very same way one shows that r^\ddagger is τ -pollable w.r.t. P . Requirement (*) w.r.t. π says that if a premise from H^d occurs in a rule r^\ddagger obtained by a subproof π^\ddagger of π , then this premise is a delayable premise of r^\ddagger as well. Any subproof of π^\ddagger can be obtained as $\rho(\pi^\ddagger)$ with π^\ddagger a subproof of π ; the rule proven by $\rho(\pi^\ddagger)$ is $\rho(r^\ddagger)$. Again, ρ renames variables occurring in the source of r^\ddagger , but leaves other variables occurring in r^\ddagger alone. Hence Lem. 9 applies to conclude that also r^\ddagger satisfies (*) w.r.t. π^\ddagger . \square

The following example shows the essentiality of limiting Lem. 11 and Lem. 14 to *linearly* provable rules. In this example two delayable positive premises in a linear ruloid are collapsed to a single premise that is not delayable in the resulting non-linear ruloid.

Example 8 Let f, g be unary, kb, kc binary, and h a ternary function symbol. Let $Act = \{a, b, c, d\}$, and consider the TSS with the following rules:

$$\begin{array}{c}
\frac{x \xrightarrow{a} y}{f(x) \xrightarrow{b} x} \quad \frac{x \xrightarrow{a} y}{f(x) \xrightarrow{c} x} \quad \frac{x \xrightarrow{\tau} y}{f(x) \xrightarrow{\tau} kb(x, y)} \quad \frac{x \xrightarrow{\tau} y}{f(x) \xrightarrow{\tau} kc(x, y)} \\
\\
\frac{x \xrightarrow{b} y \quad x \xrightarrow{c} z}{g(x) \xrightarrow{d} x} \quad \frac{x \xrightarrow{\tau} y}{g(x) \xrightarrow{\tau} h(x, y, x)} \quad \frac{x \xrightarrow{\tau} y}{g(x) \xrightarrow{\tau} h(x, x, y)} \\
\\
\frac{x_2 \xrightarrow{a} y}{kb(x_1, x_2) \xrightarrow{b} x_1} \quad \frac{x_2 \xrightarrow{\tau} y}{kb(x_1, x_2) \xrightarrow{\tau} kb(x_1, y)} \\
\\
\frac{x_2 \xrightarrow{a} y}{kc(x_1, x_2) \xrightarrow{c} x_1} \quad \frac{x_2 \xrightarrow{\tau} y}{kc(x_1, x_2) \xrightarrow{\tau} kc(x_1, y)} \\
\\
\frac{x_2 \xrightarrow{b} y \quad x_3 \xrightarrow{c} z}{h(x_1, x_2, x_3) \xrightarrow{d} x_1}
\end{array}$$

$$\frac{x_2 \xrightarrow{\tau} y}{h(x_1, x_2, x_3) \xrightarrow{\tau} h(x_1, y, x_3)} \quad \frac{x_3 \xrightarrow{\tau} y}{h(x_1, x_2, x_3) \xrightarrow{\tau} h(x_1, x_2, y)}$$

This TSS is positive and in xynft format. The argument of f and of g are taken to be \aleph -liquid and Λ -frozen. The first argument of kb , kc and h are \aleph -frozen and Λ -frozen, while their other arguments are \aleph -liquid and Λ -liquid. The TSS is $\aleph \cap \Lambda$ -patient, since there are patience rules for the latter four arguments. The rules are moreover η -bisimulation safe and satisfy condition 5 of Def. 27 w.r.t. \aleph and Λ .

The TSS is manifestly delay resistant w.r.t. Λ . To prove this, we only need to consider the rules for f and g . That the premise of the first rule for f is manifestly delay resistant follows by the rules $\frac{x \xrightarrow{\tau} z}{f(x) \xrightarrow{\tau} kb(x, z)}$ and $\frac{z \xrightarrow{a} y}{kb(x, z) \xrightarrow{b} x}$. Likewise for the second rule for f . That the premise of the third rule for f is manifestly delay resistant follows by the rules $\frac{x \xrightarrow{\tau} z}{f(x) \xrightarrow{\tau} kb(x, z)}$ and $\frac{z \xrightarrow{\tau} y}{kb(x, z) \xrightarrow{\tau} kb(x, y)}$. Likewise for the fourth rule for f . Furthermore, that the first premise of the first rule for g is manifestly delay resistant follows by the rules $\frac{x \xrightarrow{\tau} z}{g(x) \xrightarrow{\tau} h(x, z, x)}$ and $\frac{z \xrightarrow{b} y \quad x \xrightarrow{c} y'}{h(x, z, x) \xrightarrow{d} x}$. Likewise for the second premise of this rule. That the premise of the second rule for g is manifestly delay resistant follows by $\frac{x \xrightarrow{\tau} z}{g(x) \xrightarrow{\tau} h(x, z, x)}$ and $\frac{z \xrightarrow{\tau} y}{h(x, z, x) \xrightarrow{d} h(x, y, x)}$. Likewise for the third rule for g . Finally, that the TSS is manifestly delay resistant (without taking into account Λ) now follows easily by employing the four patience rules in the TSS.

The non-linear ruloid $\frac{x \xrightarrow{a} y}{g(f(x)) \xrightarrow{d} x}$ is obtained by substituting y for z in the linear ruloid $\frac{x \xrightarrow{a} y \quad x \xrightarrow{a} z}{g(f(x)) \xrightarrow{d} x}$. We argue that this non-linear ruloid is not delay resistant. Its premise $x \xrightarrow{a} y$ is not τ -pollable, as clearly $\frac{x \xrightarrow{\tau} z}{g(f(x)) \xrightarrow{d} x}$ is not a ruloid. Suppose, toward a contradiction, that it is delayable. This would mean there exist ruloids $\frac{x \xrightarrow{\tau} z}{g(f(x)) \xrightarrow{\tau} v}$ and $\frac{z \xrightarrow{a} y}{v \xrightarrow{d} x}$ for some term v and fresh variable z . The first of these two ruloids allows four possibilities for v : $h(f(x), t(x, z), f(x))$ or $h(f(x), f(x), t(x, z))$ where $t(x, z)$ equals either $kb(x, z)$ or $kc(x, z)$. However, for none of these four possibilities does there exist a ruloid $\frac{z \xrightarrow{a} y}{v \xrightarrow{d} x}$.

Proposition 12 Let P be a standard TSS in ready simulation format, in which each transition rule is rooted branching bisimulation safe and satisfies condition 5 of Def. 27 w.r.t. \aleph and Λ . Let moreover P be $\aleph \cap \Lambda$ -patient and manifestly delay resistant w.r.t. Λ . Then the standard TSS \hat{P}^\ddagger in xynft format, constructed in Secs. 2.6–2.7, is manifestly delay resistant.

Proof: By Def. 32, the conversion P^\dagger of P to decent ntyft format, defined in Sec. 2.6, is manifestly delay resistant w.r.t. Λ . By Lem. 10 it is also manifestly τ -pollable w.r.t. Λ . Hence the rules of P^\dagger are negative-stable and positive delay resistant w.r.t. Λ and P^\dagger , as well as τ -pollable w.r.t. Λ and P^\dagger . Clearly, P^\dagger is $\aleph \cap \Lambda$ -patient, and all of its rules are rooted branching bisimulation safe and satisfy condition 5 of Def. 27 w.r.t. \aleph and Λ (as already observed in the proof of Cor. 1).

The TSS \hat{P}^\ddagger contains all xynft rules linearly provable from P^\dagger . By Lem. 12 those rules are negative-stable, and by Lem. 14 they are positive delay resistant and τ -pollable w.r.t. P^\dagger . The definitions of positive delay resistant and τ -pollability of rules from \hat{P}^\ddagger w.r.t. P^\dagger require the existence of certain xynft rules that are linearly provable from P^\dagger . By definition, these are rules of \hat{P}^\ddagger . Hence \hat{P}^\ddagger is manifestly delay resistant, as well as manifestly τ -pollable. \square

B.4 Transfer of τ -pollability to non-standard rules

Lemma 15 Let P be a standard TSS in ready simulation format. If the TSS \hat{P}^\ddagger is manifestly τ -pollable and negative-stable, then its augmentation \hat{P}^+ with non-standard rules is manifestly τ -pollable.

Proof: Let $\frac{H}{t \xrightarrow{\alpha} u}$ be a rule of \hat{P}^+ , and let $M \subseteq H^+$. It suffices to show that also $\frac{H_M}{t \xrightarrow{\alpha} u}$ is a rule of \hat{P}^+ , where $H_M := (H \setminus M) \cup M_\tau$. Let $\hat{P}^\ddagger \upharpoonright (t \xrightarrow{\alpha} u)$ denote the set of rules of \hat{P}^\ddagger with a conclusion of the form $t \xrightarrow{\alpha} u$ for some u . By the construction of \hat{P}^+ there exists a surjective function f from $\hat{P}^\ddagger \upharpoonright (t \xrightarrow{\alpha} u)$ to H such that each rule $r \in \hat{P}^\ddagger \upharpoonright (t \xrightarrow{\alpha} u)$ contains a premise that denies the premise $f(r)$. It suffices to find a surjective function f_M from $\hat{P}^\ddagger \upharpoonright (t \xrightarrow{\alpha} u)$ to H_M such that each rule $r \in \hat{P}^\ddagger \upharpoonright (t \xrightarrow{\alpha} u)$ contains a premise that denies the premise $f_M(r)$. In case $f(r) \in (H \setminus M)$, we take $f_M(r) := f(r)$. In case $f(r) = (w \xrightarrow{\beta} y) \in M$, we take $f_M(r) := (w \xrightarrow{\tau} z_y) \in M_\tau$. As r must contain the premise $w \xrightarrow{\beta} y$, it also contains the premise $w \xrightarrow{\tau} z_y$, using that \hat{P}^\ddagger is negative-stable. This premise denies $f_M(r)$. Surjectivity is guaranteed by construction. \square

B.5 Lifting delay resistance to ruloids

Lemma 16 Let P be a standard TSS in ready simulation format, such that the TSS \hat{P}^\ddagger is manifestly τ -pollable. If the TSS \hat{P}^+ contains rules $\frac{H}{t \xrightarrow{\alpha} u}$ and $\frac{H^\tau}{t \xrightarrow{\tau} v}$, then it also contains a rule $\frac{H'}{t \xrightarrow{\alpha} u}$, with $H' \subseteq (H \cup H^\tau)^+ \cup (H \cup H^\tau)^{s^-}$.

Proof: Suppose \hat{P}^+ contains rules $\frac{H}{t \xrightarrow{\alpha} u}$ and $\frac{H^\tau}{t \xrightarrow{\tau} v}$. Then there exists a surjective function f from $\hat{P}^\ddagger \upharpoonright (t \xrightarrow{\alpha} u) \cup \hat{P}^\ddagger \upharpoonright (t \xrightarrow{\tau} v)$ to $H \cup H^\tau$ such that each rule $r \in \text{dom}(f)$ contains a premise that denies the premise $f(r)$. It suffices to construct a function h from $\hat{P}^\ddagger \upharpoonright (t \xrightarrow{\alpha} u)$ to $(H \cup H^\tau)^+ \cup (H \cup H^\tau)^{s^-}$ such that each rule $r \in \text{dom}(h)$ contains a premise that denies the premise $h(r)$.

Let $r = \frac{K}{t \xrightarrow{\alpha} u} \in \hat{P}^\ddagger \upharpoonright (t \xrightarrow{\alpha} u)$. Let M be the collection premises $v \xrightarrow{\beta} y$ in K for which $v \xrightarrow{\tau} z$ does not occur in $H \cup H^\tau$. Since \hat{P}^\ddagger is manifestly τ -pollable, there must be a rule $r_M = \frac{K_M}{t \xrightarrow{\alpha} u}$ or $\frac{K_M}{t \xrightarrow{\tau} w}$ in \hat{P}^\ddagger with $K_M \subseteq (K \setminus M) \cup M_\tau$. So a premise $\mu \in K_M$ denies the premise $f(r_M) \in H \cup H^\tau$. Given the definition of M , this premise μ cannot occur in M_τ . Hence $\mu \in K \setminus M$ and $f(r_M) \in (H \cup H^\tau)^+ \cup (H \cup H^\tau)^{s^-}$. Define $h(r) := f(r_M)$. \square

Lemma 17 Let P be a standard TSS in ready simulation format, such that the TSS \hat{P}^\ddagger is manifestly τ -pollable and negative-stable.

- (1) For any linear P -ruloid $\frac{H}{t \xrightarrow{\alpha} u}$ there is a linear P -ruloid $\frac{H'}{t \xrightarrow{\alpha} u}$ with $H' \subseteq H^+ \cup H^{s^-}$.
- (2) For any linear P -ruloids $\frac{H}{t \xrightarrow{\alpha} u}$ and $\frac{H^\tau}{t \xrightarrow{\tau} v}$, with t not a variable, there is a linear P -ruloid $\frac{H'}{t \xrightarrow{\alpha} u}$ with $H' \subseteq (H \cup H^\tau)^+ \cup (H \cup H^\tau)^{s^-}$.

Proof: We prove the claims by simultaneous structural induction on the linear proofs of the ruloids from \hat{P}^+ .

First consider the case that t is a variable x . Then any standard P -ruloid with source t must have the form $\frac{x \xrightarrow{\alpha} y}{x \xrightarrow{\alpha} y}$; it trivially satisfies (1).

Now let $t \notin V$. First let π be a linear proof of $\frac{H}{t \xrightarrow{\alpha} u}$. Let the decent ntyft rule r in \hat{P}^+ (and hence in \hat{P}^\ddagger) of the form

$$\frac{\{t_k \xrightarrow{\beta_k} y_k \mid k \in K\} \cup \{t_\ell \xrightarrow{\gamma_\ell} \mid \ell \in L\} \cup \{x_j \xrightarrow{\gamma_j} \mid j \in J\}}{f(x_1, \dots, x_{ar(f)}) \xrightarrow{\alpha} v}$$

and the substitution σ be used at the bottom of π . Here we require $\sigma(t_\ell) \notin V$ for all $\ell \in L$ and $\sigma(x_j) \in V$; so we have split the negative premises in two sets, depending on whether after application of σ their left-hand sides are variables. We have $\sigma(f(x_1, \dots, x_{ar(f)})) = t$ and $\sigma(v) = u$. Moreover, rules $r_k = \frac{H_k}{\sigma(t_k) \xrightarrow{\beta_k} \sigma(y_k)}$

for each $k \in K$ and $r_\ell = \frac{H_\ell}{\sigma(t_\ell) \xrightarrow{\gamma_\ell}}$ for each $\ell \in L$ are linearly provable from \hat{P}^+ by means of strict subproofs

π_k and π_ℓ of π , where $H = \bigcup_{k \in K} H_k \cup \bigcup_{\ell \in L} H_\ell \cup \{\sigma(x_j) \xrightarrow{\gamma_j} \mid j \in J\}$. Here we use that \hat{P}^+ is in ntyft format. Using that \hat{P}^\ddagger is negative-stable, $\{\sigma(x_j) \xrightarrow{\gamma_j} \mid j \in J\} \subseteq H^{s^-}$; moreover, for each $\ell \in L$ there is a unique $\ell' \in L$ with $t_{\ell'} = t_\ell$ and $\gamma_{\ell'} = \tau$; we write $H_{\ell'}^\tau := H_{\ell'}$.

As r is decent, $var(t_k) \subseteq \{x_1, \dots, x_{ar(f)}\}$, so $var(\sigma(t_k)) \subseteq var(t)$ for each $k \in K$. Likewise, $var(\sigma(t_\ell)) \subseteq var(t)$ for each $\ell \in L$. From $rhs(H) \cap var(t) = \emptyset$ it follows that $rhs(H_k) \cap var(\sigma(t_k)) = \emptyset$ for each $k \in K$, and $rhs(H_\ell) \cap var(\sigma(t_\ell)) = \emptyset$ for each $\ell \in L$. So for each $k \in K$ and $\ell \in L$, the rules r_k and r_ℓ are nxytt rules. By Lem. 1, they are decent, and thus ruloids.

By induction there is a P -ruloid $\frac{H'_k}{\sigma(t_k) \xrightarrow{\beta_k} \sigma(y_k)}$ with $H'_k \subseteq H_k^+ \cup H_k^{s^-}$, for each $k \in K$. Likewise, there

is a P -ruloid $\frac{H'_\ell}{\sigma(t_\ell) \xrightarrow{\gamma_\ell}}$ with $H'_\ell \subseteq (H_\ell \cup H_{\ell'}^\tau)^+ \cup (H_\ell \cup H_{\ell'}^\tau)^{s^-}$, for each $\ell \in L$. By composition of proofs, we

obtain a linear ruloid $\frac{H'}{t \xrightarrow{\alpha} u}$ with $H' = \bigcup_{k \in K} H'_k \cup \bigcup_{\ell \in L} H'_\ell \cup \{\sigma(x_j) \xrightarrow{\gamma_j} \mid j \in J\} \subseteq H^+ \cup H^{s^-}$.

Secondly, let π be a linear proof of $\frac{H}{t \xrightarrow{\alpha} u}$, and π^τ a linear proof of $\frac{H^\tau}{t \xrightarrow{\tau} u}$. Let the decent ntyft rule r in \hat{P}^+ (and hence in \hat{P}^\ddagger) of the form

$$\frac{\{t_k \xrightarrow{\beta_k} y_k \mid k \in K\} \cup \{t_\ell \xrightarrow{\gamma_\ell} \mid \ell \in L\} \cup \{x_j \xrightarrow{\gamma_j} \mid j \in J\}}{f(x_1, \dots, x_{ar(f)}) \xrightarrow{\alpha}}$$

and the substitution σ be used at the bottom of π , again with $\sigma(t_\ell) \notin V$ for $\ell \in L$ and $\sigma(x_j) \in V$ for $j \in J$. Likewise, let the decent ntyft rule r^τ in \hat{P}^+ of the form

$$\frac{\{t_k \xrightarrow{\beta_k} y_k \mid k \in K^\tau\} \cup \{t_\ell \xrightarrow{\gamma_\ell} \mid \ell \in L^\tau\} \cup \{x_j \xrightarrow{\gamma_j} \mid j \in J^\tau\}}{f(x_1, \dots, x_{ar(f)}) \xrightarrow{\tau}}$$

and the substitution σ^τ be used at the bottom of π^τ , with $\sigma^\tau(t_\ell) \notin V$ for $\ell \in L$ and $\sigma^\tau(x_j) \in V$ for $j \in J$. Note that $\sigma(x_i) = \sigma^\tau(x_i)$ for $i = 1, \dots, ar(f)$. By choosing the right-hand sides of premises in r^τ different from the ones in r , we may assume, w.l.o.g., that $\sigma^\tau = \sigma$.

Exactly as above, P -ruloids $r_k = \frac{H_k}{\sigma(t_k) \xrightarrow{\beta_k} \sigma(y_k)}$ for each $k \in K \cup K^\tau$ and $r_\ell = \frac{H_\ell}{\sigma(t_\ell) \xrightarrow{\gamma_\ell}}$ for each $\ell \in L \cup L^\tau$ are linearly provable from \hat{P}^+ by means of strict subproofs π_k and π_ℓ of π or π^τ , where $H = \bigcup_{k \in K} H_k \cup \bigcup_{\ell \in L} H_\ell \cup \{\sigma(x_j) \xrightarrow{\gamma_j} \mid j \in J\}$ and $H^\tau = \bigcup_{k \in K^\tau} H_k \cup \bigcup_{\ell \in L^\tau} H_\ell \cup \{\sigma(x_j) \xrightarrow{\gamma_j} \mid j \in J^\tau\}$.

By Lem. 16 there is a rule in \hat{P}^+ of the form

$$\frac{\{t_k \xrightarrow{\beta_k} y_k \mid k \in K'\} \cup \{t_\ell \xrightarrow{\gamma_\ell} \mid \ell \in L'\} \cup \{x_j \xrightarrow{\gamma_j} \mid j \in J'\}}{f(x_1, \dots, x_{ar(f)}) \xrightarrow{\alpha}}$$

with $K' \subseteq K \cup K^\tau$, $L' \subseteq L \cup L^\tau$ and $J' \subseteq J \cup J^\tau$, and such that (1) for any $\ell \in L'$ there is an $\ell' \in L \cup L^\tau$ with $t_{\ell'} = t_\ell$ and $\gamma_{\ell'} = \tau$, and (2) for any $j \in J'$ there is an $j' \in J \cup J^\tau$ with $t_{j'} = t_j$ and $\gamma_{j'} = \tau$; making an arbitrary choice for ℓ' in case of ambiguity, let $H_{\ell'}^\tau := H_{\ell'}$.

By induction there is a P -ruloid $\frac{H'_k}{\sigma(t_k) \xrightarrow{\beta_k} \sigma(y_k)}$ with $H'_k \subseteq H_k^+ \cup H_k^{s^-}$, for each $k \in K'$. Likewise, there

is a P -ruloid $\frac{H'_\ell}{\sigma(t_\ell) \xrightarrow{\gamma_\ell}}$ with $H'_\ell \subseteq (H_\ell \cup H_{\ell'}^\tau)^+ \cup (H_\ell \cup H_{\ell'}^\tau)^{s^-}$, for each $\ell \in L'$. By composition of proofs, we

obtain a linear ruloid $\frac{H'}{t \xrightarrow{\alpha} \tau}$ with $H' = \bigcup_{k \in K'} H'_k \cup \bigcup_{\ell \in L'} H'_\ell \cup \{\sigma(x_j) \xrightarrow{\gamma_j} \mid j \in J'\} \subseteq (H \cup H^\tau)^+ \cup (H \cup H^\tau)^{s-}$.
 \square

Proof of Thm. 8. Let P be a manifestly delay resistant standard TSS in ready simulation format. By Prop. 11, the TSS \hat{P}^\ddagger , constructed in Sec. 2.6, is manifestly delay resistant, and thus negative-stable. By Lem. 10, it is manifestly τ -pollable.

By Lem. 15 the TSS \hat{P}^+ is manifestly τ -pollable. So by definition the rules of \hat{P}^+ are positive delay resistant as well as τ -pollable w.r.t. \hat{P}^+ . By Lem. 11, each nxytt rule linearly provable from \hat{P}^+ , i.e. each linear P -ruloid, is positive delay resistant w.r.t. \hat{P}^+ . By Lem. 17 all linear P -ruloids are negative delay resistant w.r.t. \hat{P}^+ . Thus P is delay resistant. \square

Proof of Thm. 9. Let P be a TSS meeting the conditions of Thm. 9. By Prop. 12, the TSS \hat{P}^\ddagger is manifestly delay resistant. From here, the argument proceeds as above. \square